6

Product Mix Problems

6.1 Introduction

Product mix problems are conceptually the easiest constrained optimization problems to comprehend. The Astro/Cosmo problem considered earlier is an example. Although product mix problems are seldom encountered in their simple textbook form in practice, they very frequently constitute important components of larger problems such as multiperiod planning models.

The features of a product mix problem are that there is a collection of products competing for a finite set of resources. If there are *m* resources and *n* products, then the so-called "technology" is characterized by a table with *m* rows and *n* columns of technologic coefficients. The coefficient in row *i*, column *j*, is the number of units of resource *i* used by each unit of product *j*. The numbers in a row of the table are simply the coefficients of a constraint in the LP. In simple product mix problems, these coefficients are nonnegative. Additionally, associated with each product is a profit contribution per unit and associated with each resource is an availability. The objective is to find how much to produce of each product (i.e., the mix) to maximize profits subject to not using more of each resource than is available.

The following product mix example will illustrate not only product mix LP formulations, but also: 1) representation of nonlinear profit functions and 2) the fact that most problems have alternative correct formulations. Two people may develop different formulations of the same problem, but both may be correct.

6.2 Example

A certain plant can manufacture five different products in any combination. Each product requires time on each of three machines in the following manner (figures in minutes/unit):

	Machine		
Product	**1**	**2**	**3**
A	12	8	5
B	7	9	10
C	8	4	7
D	10	0	3
E	7	11	2

Each machine is available 128 hours per week.

Products *A*, *B,* and *C* are purely competitive and any amounts made may be sold at respective prices of $5, $4, and $5. The first 20 units of *D* and *E* produced per week can be sold at $4 each, but all made in excess of 20 can only be sold at $3 each. Variable labor costs are $4 per hour for machines 1 and 2, while machine 3 labor costs $3 per hour. Material costs are $2 for products *A* and *C*, while products *B*, *D*, and *E* only cost $1. You wish to maximize profit to the firm.

The principal complication is that the profit contributions of products *D* and *E* are not linear. You may find the following device useful for eliminating this complication. Define two additional products D_2 and E_2, which sell for $3 per unit. What upper limits must then be placed on the sale of the original products *D* and *E*? The decision variables and their profit contributions are as follows:

Decision Variables	**Definition**	**Profit Contribution per Unit**
A	Number of units of *A* produced per week	5 – 2 = $3
B	Number of units of *B* produced per week	4 – 1 = $3
C	Number of units of *C* produced per week	5 – 2 = $3
D	Number of units of *D* not in excess of 20 produced/week	$3
D_2	Number of units of *D* produced in excess of 20 per week*	$2
E	Number of units of *E* not in excess of 20 produced/week	$3
E_2	Number of units of *E* produced in excess of 20	$2
M_1	Hours of machine 1 used per week	–$4
M_2	Hours of machine 2 used per week	–$4
M_3	Hours of machine 3 used per week	–$3

*Total production of product *D* is $D + D_2$.

We will not worry about issues of sequencing the various products on each machine. This is reasonable if the due-dates for the products are far enough in the future. Our problem in this case is to:

Maximize Revenues minus costs
Subject to
Minutes used equals minutes run on each machine,
At most 20 units each can be produced of products *D* and *E*,
Each machine can be run at most 128 hours.

More precisely, the formulation in LINGO is:

```
! Maximize revenue minus costs;
MAX = 3 * A + 3 * B + 3 * C + 3 * D + 2 * D2 + 3 * E
     + 2 * E2 - 4 * M1 - 4 * M2 - 3 * M3;
! Machine time used = machine time made available;
12*A + 7*B + 8*C + 10*D + 10*D2 + 7*E +  7*E2 - 60*M1 = 0;
8*A + 9*B + 4*C + 11*E + 11*E2 - 60*M2 = 0;
5*A + 10*B + 7*C + 3*D + 3*D2 + 2*E + 2*E2 - 60*M3=0;
   D <= 20;  ! Max sellable at high price;
   E <= 20;
!Machine availability;
   M1 <= 128;
   M2 <= 128;
   M3 <= 128;
END
```

The first three constraints have the units of "minutes" and specify the hours of machine time as a function of the number of units produced. The next two constraints place upper limits on the number of high profit units of *D* and *E* that may be sold. The final three constraints put upper limits on the amount of machine time that may be used and have the units of "hours".

Constraint 2 can be first written as:

$$\frac{12A + 7B + 8C + 10D + 10D_2 + 7E + 7E_2}{60} = M_1$$

Multiplying by 60 and bringing M_1 to the left gives the second constraint. The solution is:

```
Optimal solution found at step:         4
Objective value:                 1777.625
Variable           Value        Reduced Cost
       A       0.0000000            1.358334
       B       0.0000000           0.1854168
       C        942.5000           0.0000000
       D       0.0000000           0.1291668
      D2       0.0000000            1.129167
       E        20.00000           0.0000000
      E2       0.0000000           0.9187501
      M1        128.0000           0.0000000
      M2        66.50000           0.0000000
      M3        110.6250           0.0000000
```

```
Row     Slack or Surplus       Dual Price
  1        1777.625             1.000000
  2       0.0000000            0.2979167
  3       0.0000000            0.6666667E-01
  4       0.0000000            0.5000000E-01
  5        20.00000            0.0000000
  6       0.0000000            0.8125000E-01
  7       0.0000000             13.87500
  8        61.50000            0.0000000
  9        17.37500            0.0000000
```

The form of the solution is quite simple to state: make as many of E as possible (20). After that, make as much of product C as possible until we run out of capacity on machine 1.

This problem is a good example of one for which it is very easy to develop alternative formulations of the same problem. These alternative formulations are all correct, but may have more or less constraints and variables. For example, the constraint:

$$8A + 9B + 4C + 11E + 11E_2 - 60M_2 = 0$$

can be rewritten as:

$$M_2 = (8A + 9B + 4C + 11E + 11E_2)/60.$$

The expression on the right-hand side can be substituted for M_2 wherever M_2 appears in the formulation. Because the expression on the right-hand side will always be nonnegative, the nonnegativity constraint on M_2 will automatically be satisfied. Thus, M_2 and the above constraint can be eliminated from the problem if we are willing to do a bit of arithmetic. When similar arguments are applied to M_1 and M_3 and the implied divisions are performed, one obtains the formulation:

```
MAX = 1.416667*A + 1.433333*B + 1.85*C + 2.183334*D + 1.183333*D2 +
1.7*E + .7*E2;
! Machine time used = machine time made available;
12*A + 7*B + 8*C + 10*D + 10*D2 + 7*E + 7*E2 <= 7680;
8*A + 9*B + 4*C + 11*E + 11*E2 <= 7680;
5*A + 10*B + 7*C + 3*D + 3*D2 + 2*E + 2*E2 <= 7680;
! Product limits;
D < 20;
E < 20;
```

This looks more like a standard product mix formulation. All the constraints are capacity constraints of some sort. Notice the solution to this formulation is really the same as the previous formulation:

```
Optimal solution found at step:          6
Objective value:                  1777.625
Variable          Value        Reduced Cost
       A      0.0000000            1.358333
       B      0.0000000           0.1854170
       C       942.5000           0.0000000
       D      0.0000000           0.1291660
      D2      0.0000000            1.129167
       E       20.00000           0.0000000
      E2      0.0000000           0.9187500
     Row    Slack or Surplus     Dual Price
       1       1777.625            1.000000
       2      0.0000000           0.2312500
       3       3690.000           0.0000000
       4       1042.500           0.0000000
       5       20.00000           0.0000000
       6      0.0000000           0.8125000E-01
```

The lazy formulator might give the first formulation, whereas the second formulation might be given by the person who enjoys doing arithmetic.

6.3 Process Selection Product Mix Problems

A not uncommon feature of product mix models is two or more distinct variables in the LP formulation may actually correspond to alternate methods for producing the same product. In this case, the LP is being used not only to discover how much should be produced of a product, but also to select the best process for producing each product.

A second feature that usually appears with product mix problems is a requirement that a certain amount of a product be produced. This condition takes the problem out of the realm of simple product mix. Nevertheless, let us consider a problem with the above two features.

The American Metal Fabricating Company (AMFC) produces various products from steel bars. One of the initial steps is a shaping operation performed by rolling machines. There are three machines available for this purpose, the B_3, B_4, and B_5. The following table gives their features:

Machine	Speed in Feet per Minute	Allowable Raw Material Thickness in Inches	Available Hours per Week	Labor Cost Per Hour Operating
B_3	150	3/16 to 3/8	35	$10
B_4	100	5/16 to 1/2	35	$15
B_5	75	3/8 to 3/4	35	$17

This kind of combination of capabilities is not uncommon. That is, machines that process larger material operate at slower speed.

This week, three products must be produced. AMFC must produce at least 218,000 feet of $\frac{1}{4}$" material, 114,000 feet of $\frac{3}{8}$" material, and 111,000 feet of $\frac{1}{2}$" material. The profit contributions per foot excluding labor for these three products are 0.017, 0.019, and 0.02. These prices apply to all production (e.g., any in excess of the required production). The shipping department has a capacity limit of 600,000 feet per week, regardless of the thickness.

What are the decision variables and constraints for this problem? The decision variables require some thought. There is only one way of producing ¼" material, three ways of producing ⅜", and two ways of producing ½". Thus, you will want to have at least the following decision variables. For numerical convenience, we measure length in thousands of feet:

B_{34} = 1,000's of feet of ¼" produced on B_3,
B_{38} = 1,000's of feet of ⅜" produced on B_3,
B_{48} = 1,000's of feet of ⅜" produced on B_4,
B_{58} = 1,000's of feet of ⅜" produced on B_5,
B_{42} = 1,000's of feet of ½" produced on B_4,
B_{52} = 1,000's of feet of ½" produced on B_5.

For the objective function, we must have the profit contribution including labor costs. When this is done, we obtain:

Variable	Profit Contribution per Foot
B_{34}	0.01589
B_{38}	0.01789
B_{48}	0.01650
B_{58}	0.01522
B_{42}	0.01750
B_{52}	0.01622

Clearly, there will be four constraints corresponding to AMFC's three scarce machine resources and its shipping department capacity. There should be three more constraints due to the production requirements in the three products. For the machine capacity constraints, we want the number of hours required for 1,000 feet processed. For machine B_3, this figure is 1,000/(60 min./hr.) × (150 ft./min.) = 0.111111 hours per 1,000 ft. Similar figures for B_4 and B_5 are 0.16667 hours per 1,000 ft. and 0.22222 hours per 1,000 feet.

The formulation can now be written:

$$\text{Maximize} = 15.89B_{34} + 17.89B_{38} + 16.5B_{48} + 15.22B_{58} + 17.5B_{42} + 16.22B_{52}$$

subject to

$0.11111B_{34} + 0.11111B_{38} \leq 35$ *Machine*
$0.16667B_{48} + 0.16667B_{42} \leq 35$ *capacities*
$0.22222B_{58} + 0.22222B_{52} \leq 35$ *in hours*
$B_{34} + B_{38} + B_{48} + B_{58} + B_{42} + B_{52} \leq 600$ *Shipping capacity in 1,000's of feet*
$B_{34} \geq 218$ *Production*
$B_{38} + B_{48} + B_{58} \geq 114$ *requirements*
$B_{42} + B_{52} \geq 111$ *in 1,000's of feet*

Without the last three constraints, the problem is a simple product mix problem.

It is a worthwhile exercise to attempt to deduce the optimal solution just from cost arguments. The $\frac{1}{4}$" product can be produced on only machine B_3, so we know B_{34} is at least 218. The $\frac{3}{8}$" product is more profitable than the $\frac{1}{4}$" on machine B_3. Therefore, we can conclude that $B_{34} = 218$ and B_{38} will take up the slack. The $\frac{1}{2}$" and the $\frac{3}{8}$" product can be produced on either B_4 or B_5. In either case, the $\frac{1}{2}$" is more profitable per foot, so we know B_{48} and B_{58} will be no greater than absolutely necessary. The question is: What is "absolutely necessary"? The $\frac{3}{8}$" is more profitably run on B_3 than on B_4 or B_5. Therefore, it follows that we will satisfy the $\frac{3}{8}$" demand from B_3 and, if sufficient, the remainder from B_4 and then from B_5. Specifically, we proceed as follows:

Set $B_{34} = 218$.

This leaves a slack of $35 - 218 \times 0.11111 = 10.78$ hours on B_3. This is sufficient to produce 97,000 feet of $\frac{3}{8}$", so we conclude that:

$B_{38} = 97$.

The remainder of the $\frac{3}{8}$" demand must be made up from either machine B_4 or B_5. It would appear that it should be done on machine B_4 because the profit contribution for $\frac{3}{8}$" is higher on B_4 than B_5. Note, however, that $\frac{1}{2}$" is also more profitable on B_4 than B_5 by exactly the same amount. Thus, we are indifferent. Let us arbitrarily use machine B_4 to fill the rest of $\frac{3}{8}$" demand. Thus:

$B_{48} = 17$.

Now, any remaining capacity will be used to produce $\frac{1}{2}$" product. There are $35 - 17 \times 0.16667 = 32.16667$ hours of capacity on B_4. At this point, we should worry about shipping capacity. We still have capacity for $600 - 218 - 97 - 17 = 268$ in 1,000's of feet. B_{42} is more profitable than B_{52}, so we will make it as large as possible. Namely, $32.16667/0.16667 = 193$, so:

$B_{42} = 193$.

The remaining shipping capacity is $268 - 193 = 75$, so:

$B_{52} = 75$.

Any LP is in theory solvable by similar manual economic arguments, but the calculations could be very tedious and prone to errors of both arithmetic and logic. If we take the lazy route and solve it with LINGO, we get the same solution as our manual one:

```
Optimal solution found at step:         2
Objective value:                 10073.85
Variable          Value        Reduced Cost
     B34       218.00000         0.000000
     B38        97.00315         0.000000
     B48        16.99685         0.000000
     B58         0.00000         0.000000
     B42       192.99900         0.000000
     B52        75.00105         0.000000
     Row    Slack or Surplus      Dual Price
       1        10073.85         1.000000
       2        0.000000        24.030240
       3        0.000000         7.679846
       4       18.333270         0.000000
       5        0.000000        16.220000
       6        0.000000        -3.000000
       7        0.000000        -1.000000
       8      157.000000         0.000000

Ranges in which the basis is unchanged:
                     Objective Coefficient Ranges
                Current       Allowable      Allowable
Variable    Coefficient        Increase       Decrease
     B34       15.89000        3.000000       INFINITY
     B38       17.89000        INFINITY       2.670000
     B48       16.50000        1.000000            0.0
     B58       15.22000        0.000000       INFINITY
     B42       17.50000             0.0       1.000000
     B52       16.22000        1.280000            0.0
                          Right-hand Side Ranges
     Row        Current       Allowable      Allowable
                    RHS        Increase       Decrease
       2       35.00000        1.888520       9.166634
       3       35.00000        12.50043       13.75036
       4       35.00000        INFINITY       18.33327
       5       600.0000        82.50053       75.00105
       6       218.0000        97.00315       16.99685
       7       114.0000        157.0000       16.99685
       8       111.0000        157.0000       INFINITY
```

Notice *B58* is zero, but its reduced cost is also zero. This means *B58* could be increased (and *B48* decreased) without affecting profits. This is consistent with our earlier statement that we were indifferent between using *B48* and *B58* to satisfy the $\frac{3}{8}$" demand.

Below is a sets version of the problem:

```
!This is a sets version of the previous example;
MODEL:
SETS:
   MACHINE / B3, B4, B5 / : HPERWK, TIME;
!This is the coefficient for the time per day constraint;
   THICKNESS / FOURTH, EIGHT, HALF / : NEED;
!This is the amount of each thickness needed
to be produced;
   METHOD ( MACHINE, THICKNESS ) : VOLUME, PROFIT, POSSIBLE;
!VOLUME is the variable, PROFIT the objective coefficients, and POSSIBLE
is a Boolean representing whether it is possible to produce the given
thickness;
ENDSETS
DATA:
!  Hours/week available on each machine;
    HPERWK =   35,    35,    35;
! Hours per 1000 feet for each machine;
      TIME = .11111 .16667 .22222;
! Amount needed of each product;
      NEED = 218     114     111;
! Profit by product and machine;
    PROFIT = 15.89, 17.89,  0,
              0,     16.5,  17.5,
              0,     15.22, 16.22;
! Which products can be made on which machine;
  POSSIBLE =  1,      1,      0,
              0,      1,      1,
              0,      1,      1;
! Shipping capacity per day;
  SHPERDAY = 600;
ENDDATA
!--------------------------------------------------;
!Objective function;
MAX = @SUM( METHOD(I,J): VOLUME(I,J)  * PROFIT(I,J));
@SUM( METHOD( K, L): VOLUME( K, L)) <= SHPERDAY;
!This is the max amount that can be made each day;
@FOR( MACHINE( N):
 ! Maximum time each machine can be used/week.;
 @SUM( THICKNESS( M):
   POSSIBLE(N,M)  * VOLUME(N,M)  * TIME(N))<=HPERWK(N);););
 @FOR( THICKNESS( Q)  :
 !Must meet demand for each thickness;
 @SUM( MACHINE(P): POSSIBLE(P,Q)*VOLUME(P,Q))>=NEED(Q));
END
```

6.4 Problems

1. Consider a manufacturer that produces two products, Widgets and Frisbees. Each product is made from the two raw materials, polyester and polypropylene. The following table gives the amounts required of each of the two products:

Widgets	Frisbees	Raw Material
3	5	Polyester
6	2	Polypropylene

Because of import quotas, the company is able to obtain only 12 units and 10 units of polyester and polypropylene, respectively, this month. The company is interested in planning its production for the next month. For this purpose, it is important to know the profit contribution of each product. These contributions have been found to be \$3 and \$4 for Widgets and Frisbees, respectively. What should be the amounts of Widgets and Frisbees produced next month?

2. The Otto Maddick Machine Tool Company produces two products, muffler bearings and torque amplifiers. One muffler bearing requires $1/8$ hour of assembly labor, 0.25 hours in the stamping department, and 9 square feet of sheet steel. Each torque amplifier requires $1/3$ hour in both assembly and stamping and uses 6 square feet of sheet steel. Current weekly capacities in the two departments are 400 hours of assembly labor and 350 hours of stamping capacity. Sheet steel costs 15 cents per square foot. Muffler bearings can be sold for \$8 each. Torque amplifiers can be sold for \$7 each. Unused capacity in either department cannot be laid off or otherwise fruitfully used.

   a) Formulate the LP useful in maximizing the weekly profit contribution.
   b) It has just been discovered that two important considerations were not included.
      i. Up to 100 hours of overtime assembly labor can be scheduled at a cost of \$5 per hour.
      ii. The sheet metal supplier only charges 12 cents per square foot for weekly usage in excess of 5000 square feet.

   Which of the above considerations could easily be incorporated in the LP model and how? If one or both cannot be easily incorporated, indicate how you might nevertheless solve the problem.

3. Review the solution to the 5-product, 3-machine product mix problem introduced at the beginning of the chapter.

   a) What is the marginal value of an additional hour of capacity on each of the machines?
   b) The current selling price of product *A* is \$5. What would the price have to be before we would produce any *A*?
   c) It would be profitable to sell more of product *E* at \$4 if you could, but it is not profitable to sell *E* at \$3 per unit even though you can. What is the breakeven price at which you would be indifferent about selling any more *E*?
   d) It is possible to gain additional capacity by renting by the hour automatic versions of each of the three machines. That is, they require no labor. What is the maximum hourly rate you would be willing to pay to rent each of the three types of automatic machines?

4. The Aviston Electronics Company manufactures motors for toys and small appliances. The marketing department is predicting sales of 6,100 units of the Dynamonster motor in the next quarter. This is a new high and meeting this demand will test Aviston's production capacities. A Dynamonster is assembled from three components: a shaft, base, and cage. It is clear that some of these components will have to be purchased from outside suppliers because of limited in-house capacity. The variable in-house production cost per unit is compared with the outside purchase cost in the following table.

Component	Outside Cost	Inside Cost
Shaft	1.21	0.81
Base	2.50	2.30
Cage	1.95	1.45

Aviston's plant consists of three departments. The time requirements in hours of each component in each department if manufactured in-house are summarized in the following table. The hours available for Dynamonster production are listed in the last row.

Component	Cutting Department	Shaping Department	Fabrication Department
Shaft	0.04	0.06	0.04
Base	0.08	0.02	0.05
Cage	0.07	0.09	0.06
Capacity	820	820	820

a) What are the decision variables?

b) Formulate the appropriate LP.

c) How many units of each component should be purchased outside?

5. Buster Sod's younger brother, Marky Dee, operates three ranches in Texas. The acreage and irrigation water available for the three farms are shown below:

Farm	Acreage	Water Available (acre feet)
1	400	1500
2	600	2000
3	300	900

Three crops can be grown. However, the maximum acreage that can be grown of each crop is limited by the amount of appropriate harvesting equipment available. The three crops are described below:

Crop	Total Harvesting Capacity (in acres)	Water Requirements (in acre-feet/acre)	Expected Profit (in $/acre)
Milo	700	6	400
Cotton	800	4	300
Wheat	300	2	100

Any combination of crops may be grown on a farm.

a) What are the decision variables?

b) Formulate the LP.

6. Review the formulation and solution of the American Metal Fabricating process selection/product mix problem in this chapter. Based on the solution report:

    a) What is the value of an additional hour of capacity on the B_4 machine?

    b) What is the value of an additional 2 hours of capacity on the B_3 machine?

    c) By how much would one have to raise the profit contribution/1,000 ft. of $\frac{1}{4}$" material before it would be worth producing more of it?

    d) If the speed of machine B_5 could be doubled without changing the labor cost, what would it be worth per week? (Note labor on B_5 is \$17/hour.)

7. A coupon recently appeared in an advertisement in the weekend edition of a newspaper. The coupon provided \$1 off the price of any size jar of Ocean Spray cranberry juice. The cost of the weekend paper was more than \$1.

    Upon checking at a local store, we found two sizes available as follows:

Size in oz.	Price	Price/oz. w/o Coupon	Price/oz. with Coupon
32	2.09	.0653125	.0340625
48	2.89	.0602083	.039375

What questions, if any, should we ask in deciding which size to purchase? What should be our overall objective in analyzing a purchasing decision such as this?