11

Formulating and Solving Integer Programs

"To be or not to be" is true.
-G. Boole

11.1 Introduction

In many applications of optimization, one would really like the decision variables to be restricted to integer values. One is likely to tolerate a solution recommending GM produce 1,524,328.37 Chevrolets. No one will mind if this recommendation is rounded up or down. If, however, a different study recommends the optimum number of aircraft carriers to build is 1.37, then a lot of people around the world will be very interested in how this number is rounded. It is clear the validity and value of many optimization models could be improved markedly if one could restrict selected decision variables to integer values.

All good commercial optimization modeling systems are augmented with a capability that allows the user to restrict certain decision variables to integer values. The manner in which the user informs the program of this requirement varies from program to program. In LINGO, for example, one way of indicating variable X is to be restricted to integer values is to put it in the model the declaration as: @GIN(X). The important point is it is straightforward to specify this restriction. We shall see later that, even though easy to specify, sometimes it may be difficult to solve problems with this restriction. The methods for formulating and solving problems with integrality requirements are called *integer programming*.

11.1.1 Types of Variables

One general classification is according to types of variables:

Pure vs. mixed. In a pure integer program, all variables are restricted to integer values. In a mixed formulation, only certain of the variables are integer; whereas, the rest are allowed to be continuous.

0/1 vs. general. In many applications, the only integer values allowed are 0/1. Therefore, some integer programming codes assume integer variables are restricted to the values 0 or 1.

The integrality enforcing capability is perhaps more powerful than the reader at first realizes. A frequent use of integer variables in a model is as a zero/one variable to represent a go/no-go decision. It is probably true that the majority of real-world integer programs are of the zero/one variety.

11.2 Exploiting the IP Capability: Standard Applications

You will frequently encounter LP problems with the exception of just a few combinatorial complications. Many of these complications are fairly standard. The next several sections describe many of the standard complications along with the methods for incorporating them into an IP formulation. Most of these complications only require the 0/1 capability rather than the general integer capability. Binary variables can be used to represent a wide variety of go/no-go, or make-or-buy decisions. In the latter use, they are sometimes referred to as "Hamlet" variables as in: "To buy or not to buy, that is the question". Binary variables are sometimes also called Boolean variables in honor of the logician George Boole. He developed the rules of the special algebra, now known as Boolean algebra, for manipulating variables that can take on only two values. In Boole's case, the values were "True" and "False". However, it is a minor conceptual leap to represent "True" by the value 1 and "False" by the value 0. The power of these methods developed by Boole is undoubtedly the genesis of the modern compliment: "Strong, like Boole."

11.2.1 Binary Representation of General Integer Variables

Some algorithms apply to problems with only 0/1 integer variables. Conceptually, this is no limitation, as any general integer variable with a finite range can be represented by a set of 0/1 variables. For example, suppose X is restricted to the set [0, 1, 2,...,15]. Introduce the four 0/1 variables: y_1, y_2, y_3, and y_4. Replace every occurrence of X by $y_1 + 2 \times y_2 + 4 \times y_3 + 8 \times y_4$. Note every possible integer in [0, 1, 2, ..., 15] can be represented by some setting of the values of y_1, y_2, y_3, and y_4. Verify that, if the maximum value X can take on is 31, you will need 5 0/1 variables. If the maximum value is 63, you will need 6 0/1 variables. In fact, if you use k 0/1 variables, the maximum value that can be represented is 2^k-1. You can write: $VMAX = 2^k$-1. Taking logs, you can observe that the number of 0/1 variables required in this so-called binary expansion is approximately proportional to the log of the maximum value X can take on.

Although this substitution is valid, it should be avoided if possible. Most integer programming algorithms are not very efficient when applied to models containing this substitution.

11.2.2 Minimum Batch Size Constraints

When there are substantial economies of scale in undertaking an activity regardless of its level, many decision makers will specify a minimum "batch" size for the activity. For example, a large brokerage firm may require that, if you buy any bonds from the firm, you must buy at least 100. A zero/one variable can enforce this restriction as follows. Let:

x = activity level to be determined (e.g., no. of bonds purchased),
y = a zero/one variable = 1, if and only if $x > 0$,
B = minimum batch size for x (e.g., 100), and
U = known upper limit on the value of x.

The following two constraints enforce the minimum batch size condition:

$$x \leq Uy$$
$$By \leq x.$$

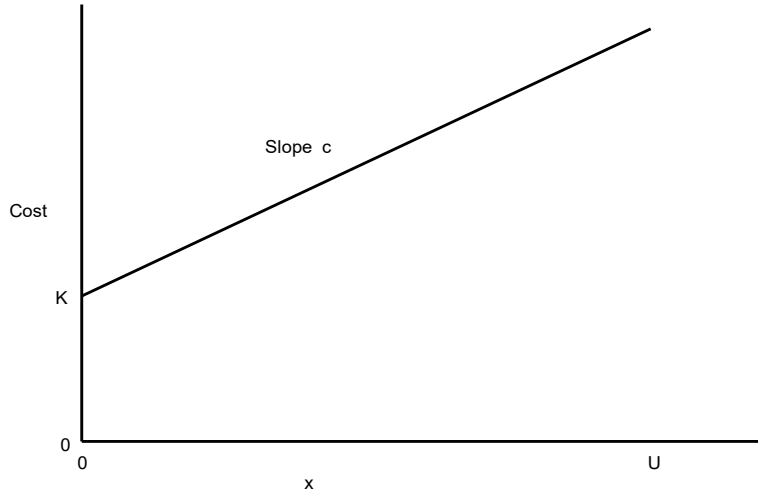
If $y = 0$, then the first constraint forces $x = 0$. While, if $y = 1$, the second constraint forces x to be at least B. Thus, y acts as a switch, which forces x to be either 0 or greater than B. The constant U should be chosen with care. For reasons of computational efficiency, it should be as small as validly possible.

Some IP packages allow the user to directly represent minimum batch size requirements by way of so-called semi-continuous variables. A variable x is semi-continuous if it is either 0 or in the range $B \le x \le \infty$. No binary variable need be explicitly introduced.

11.2.3 Fixed Charge Problems

A situation closely related to the minimum batch size situation is one where the cost function for an activity is of the fixed plus linear type indicated in Figure 11.1:

Figure 11.1 A Fixed Plus Linear Cost Curve



Define x, y, and U as before, and let K be the fixed cost incurred if $x > 0$. Then, the following components should appear in the formulation:

Minimize $Ky + cx + \ldots$
subject to
$x \le Uy$
.
.
.

The constraint and the term Ky in the objective imply x cannot be greater than 0 unless a cost K is incurred. Again, for computational efficiency, U should be as small as validly possible.

11.2.4 The Simple Plant Location Problem

The Simple Plant Location Problem (SPL) is a commonly encountered form of fixed charge problem. It is specified as follows:

n = the number of sites at which a plant may be located or opened,
m = the number of customer or demand points, each of which must be assigned to a plant,
k = the number of plants which may be opened,
f_i = the fixed cost (e.g., per year) of having a plant at site i, for $i = 1, 2, \ldots, n$,
c_{ij} = cost (e.g., per year) of assigning customer j to a plant at site i, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

The goal is to determine the set of sites at which plants should be located and which site should service each customer.

A situation giving rise to the SPL problem is the lockbox location problem encountered by a firm with customers scattered over a wide area. The plant sites, in this case, correspond to sites at which the firm might locate a postal lockbox that is managed by a bank at the site. The customer points would correspond to the, 100 say, largest metropolitan areas in the firm's market. A customer would mail his or her monthly payments to the closest lockbox. The reason for resorting to multiple lockboxes rather than having all payments mailed to a single site is several days of mail time may be saved. Suppose a firm receives \$60 million per year through the mail. The yearly cost of capital to the firm is 10% per year, and it could reduce the mail time by two days. This reduction has a yearly value of about \$30,000.

The f_i for a particular site would equal the yearly cost of having a lockbox at site i regardless of the volume processed through the site. The cost term c_{ij} would approximately equal the product: (daily cost of capital) × (mail time in days between i and j) × (yearly dollar volume mailed from area j).

Define the decision variables:

y_i = 1 if a plant is located at site i, else 0,
x_{ij} = 1 if the customer j is assigned to a plant site i, else 0.

A compact formulation of this problem as an IP is:

$$\text{Minimize} \quad \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1 \text{ to } m, \tag{2}$$

$$\sum_{j=1}^{m} x_{ij} \le m y_i \quad \text{for } i = 1 \text{ to } n, \tag{3}$$

$$\sum_{i=1}^{n} y_i = k, \tag{4}$$

$$y_i = 0 \text{ or } 1 \quad \text{for } i = 1 \text{ to } n, \tag{5}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for } i = 1 \text{ to } n, j = 1 \text{ to } m. \tag{6}$$

The constraints in (2) force each customer j to be assigned to exactly one site. The constraints in (3) force a plant to be located at site i if any customer is assigned to site i.

The reader should be cautioned against trying to solve a problem formulated in this fashion because the solution process may require embarrassingly much computer time for all, but the smallest problem. The difficulty arises because, when the problem is solved as an LP (i.e., with the conditions in (5) and (6) deleted), the solution tends to be highly fractional and with little similarity to the optimal IP solution.

A "tighter" formulation, which frequently produces an integer solution naturally when solved as an LP, is obtained by replacing (3) by the formula:

$$x_{ij} \le y_i \text{ for } i = 1 \text{ to } n, j = 1 \text{ to } m. \tag{3'}$$

At first glance, replacing (3) by (3') may seem counterproductive. If there are 20 possible plant sites and 60 customers, then the set (3) would contain 20 constraints, whereas set (3') would contain 20 × 60 = 1,200 constraints. Empirically, however, it appears to be the rule rather than the exception that, when the problem is solved as an LP with (3') rather than (3), the solution is naturally integer.

11.2.5 The Capacitated Plant Location Problem (CPL)

The CPL problem arises from the SPL problem if the volume of demand processed through a particular plant is an important consideration. In particular, the CPL problem assumes each customer has a known volume and each plant site has a known volume limit on total volume assigned to it. The additional parameters to be defined are:

D_j = volume or demand associated with customer j,
K_i = capacity of a plant located at i

The IP formulation is:

$$\text{Minimize} \quad \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \tag{7}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1 \text{ to } m \tag{8}$$

$$\sum_{j=1}^{m} D_j x_{ij} \le K_i y_i \qquad \text{for } i = 1 \text{ to } n \tag{9}$$

$$x_{ij} \le y_i \qquad \text{for } i = 1 \text{ to } n, j = 1 \text{ to } m. \tag{10}$$

$$y_i = 0 \text{ or } 1 \qquad \text{for } i = 1 \text{ to } n \tag{11}$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for } i = 1 \text{ to } n, j = 1 \text{ to } m. \tag{12}$$

This is the "single-sourcing" version of the problem. Because the variables x_{ij} are restricted to 0 or 1, each customer must have all of its volume assigned to a single plant. If "split-sourcing" is allowed, then the variables x_{ij} are allowed to be fractional with the interpretation that x_{ij} is the fraction of customer j's volume assigned to plant site i. In this case, condition (12) is dropped. Split sourcing, considered alone, is usually undesirable. An example is the assignment of elementary schools to high schools. Students who went to the same elementary school prefer to be assigned to the same high school.

Example: Capacitated Plant Location

Some of the points just mentioned will be illustrated with the following example.

The Zzyzx Company of Zzyzx, California currently has a warehouse in each of the following cities: (*A*) Baltimore, (*B*) Cheyenne, (*C*) Salt Lake City, (*D*) Memphis, and (*E*) Wichita. These warehouses supply customer regions throughout the U.S. It is convenient to aggregate customer areas and consider the customers to be located in the following cities: (*1*) Atlanta, (*2*) Boston, (*3*) Chicago, (*4*) Denver, (*5*) Omaha, and (*6*) Portland, Oregon. There is some feeling that Zzyzx is "overwarehoused". That is, it may be able to save substantial fixed costs by closing some warehouses without unduly increasing transportation and service costs. Relevant data has been collected and assembled on a "per month" basis and is displayed below:

Cost per Ton-Month Matrix

	Demand City						Monthly Supply Capacity in Tons	Monthly Fixed Cost
Warehouse	1	2	3	4	5	6		
A	$1675	$400	$685	$1630	$1160	$2800	18	$7,650
B	1460	1940	970	100	495	1200	24	3,500
C	1925	2400	1425	500	950	800	27	3,500
D	380	1355	543	1045	665	2321	22	4,100
E	922	1646	700	508	311	1797	31	2,200
Monthly Demand in Tons	10	8	12	6	7	11		

For example, closing the warehouse at *A* (Baltimore) would result in a monthly fixed cost saving of \$7,650. If *5* (Omaha) gets all of its monthly demand from *E* (Wichita), then the associated transportation cost for supplying Omaha is $7 \times 311 = \$2{,}177$ per month. A customer need not get all of its supply from a single source. Such "multiple sourcing" may result from the limited capacity of each warehouse (e.g., Cheyenne can only process 24 tons per month. Should Zzyzx close any warehouses and, if so, which ones?)

We will compare the performance of four different methods for solving, or approximately solving, this problem:

1) Loose formulation of the IP.
2) Tight formulation of the IP.
3) Greedy open heuristic: start with no plants open and sequentially open the plant giving the greatest reduction in cost until it is worthless to open further plants.
4) Greedy close heuristic: start with all plants open and sequentially close the plant saving the most money until it is worthless to close further plants.

The advantage of heuristics 3 and 4 is they are easy to apply. The performance of the four methods is as follows:

Method	Objective value: Best Solution	Computing Time in Seconds	Plants Open	Objective value: LP Solution
Loose IP	46,031	3.38	A,B,D	35,662
Tight IP	46,031	1.67	A,B,D	46,031
Greedy Open Heuristic	46,943	nil	A,B,D,E	—
Greedy Close Heuristic	46,443	nil	A,C,D,E	—

Notice, even though the loose IP finds the same optimum as the tight formulation (as it must), it takes about twice as much computing time. For large problems, the difference becomes much more dramatic. Notice for the tight formulation, however, the objective function value for the LP solution is the same as for the IP solution. When the tight formulation was *solved as an LP*, the solution was naturally integer.

The single product dynamic lotsizing problem is described by the following parameters:

n = number of periods for which production is to be planned for a product;
D_j = predicted demand in period j, for $j = 1, 2, \ldots, n$;
f_i = fixed cost of making a production run in period i;
h_i = cost per unit of product carried from period i to $i + 1$.

This problem can be cast as a simple plant location problem if we define:

$$c_{ij} = D_j \sum_{t=i}^{j-1} h_t.$$

That is, c_{ij} is the cost of supplying period j's demand from period i production. Each period can be thought of as both a potential plant site (period for a production run) and a customer.

If, further, there is a finite production capacity, K_i, in period i, then this capacitated dynamic lotsizing problem is a special case of the capacitated plant location problem.

Dual Prices and Reduced Costs in Integer Programs
Dual prices and reduced costs in solution reports for integer programs have a restricted interpretation. For first time users of IP, it is best to simply disregard the reduced cost and dual price column in the solution report. For the more curious, the dual prices and reduced costs in a solution report are obtained from the linear program that remains after all integer variables have been fixed at their optimal values and removed from the model. Thus, for a pure integer program (i.e., all variables are required to be integer), you will generally find:

- all dual prices are zero, and
- the reduced cost of a variable is simply its objective function coefficient (with sign reversed if the objective is MAX).

For mixed integer programs, the dual prices may be of interest. For example, for a plant location problem where the location variables are required to be integer, but the quantity-shipped variables are continuous, the dual prices reported are those from the continuous problem where the locations of plants have been specified beforehand (at the optimal locations).

11.2.6 Modeling Alternatives with the Scenario Approach

We may be confronted by alternatives in two different ways: a) we have to choose among two or more alternatives and we want to figure out which is best, or b) nature or the market place will choose one of two or more alternatives, and we are not sure which alternative nature will choose, so we want to analyze all alternatives so we will be prepared to react optimally once we learn which alternative was chosen by nature. Here we consider only situation (a). We call the approach the scenario approach or the disjunctive formulation, see for example Balas(1979) or section 16.2.3 of Martin(1999).

Suppose that if we disregard the alternatives, our variables are simply called $x_1, x_2, \ldots, x_n$. We call the conditions that must hold if alternative s is chosen, scenario s. Without too much loss of generality, we assume all variables are non-negative. The scenario/disjunctive approach to formulating a discrete decision problem proceeds as follows:

For each scenario s:

1) Write all the constraints that must hold if scenario s is chosen.
2) For all variables in these constraints add a subscript s, to distinguish them from equivalent variables in other scenarios. So x_j in scenario s becomes x_{sj}.
3) Add a 0/1 variable, y_s, to the model with the interpretation that $y_s = 1$ if scenario s is chosen, else 0.
4) Multiply the RHS constant term of each constraint in scenario s by y_s.
5) For each variable x_{sj} that appears in any of the scenario s constraints, add the constraint:
   $x_{sj} \le M^* y_s$, where M is a large positive constant. The purpose of this step is to force all variables in scenario s to be 0 if scenario s is not chosen.

Finally, we tie all the scenarios together with:

$\sum_s y_s = 1$, i.e., we must choose one scenario;

For each variable x_j, add the constraint:

$x_j = \sum_s x_{sj}$, so x_j takes on the value appropriate to which scenario was chosen.

For example, if just after step 1 we had a constraint of the form:

$\sum_j a_{sj}*x_j \le a_{s0}$,

then steps 2-4 would convert it to:

$\sum_j a_{sj}*x_{sj} \le a_{s0}*y_s$,

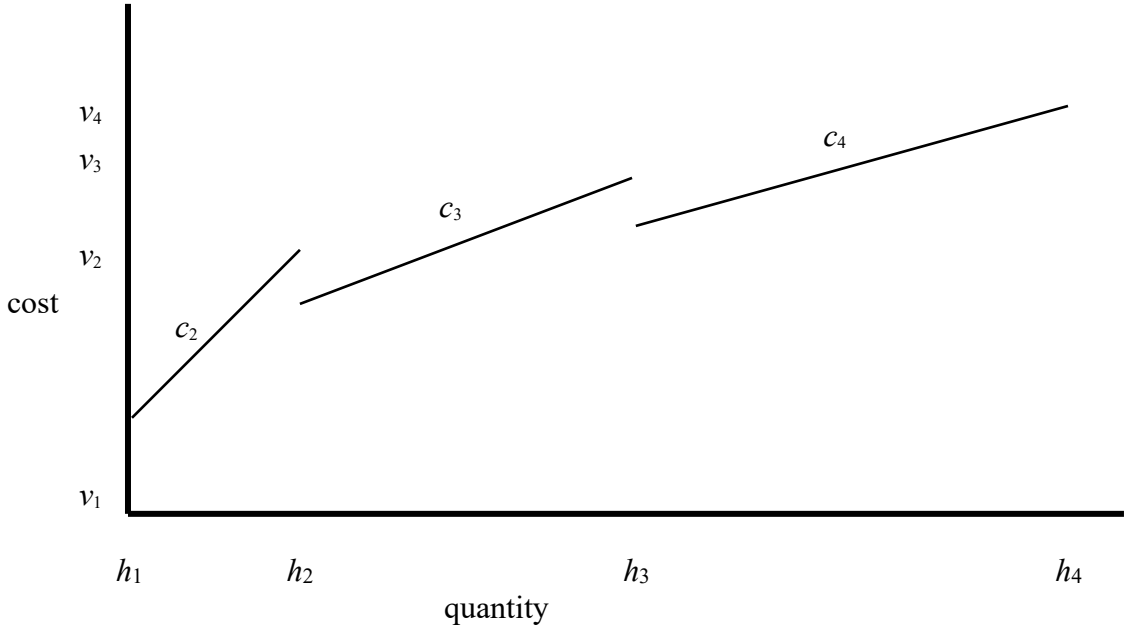
The forcing constraints in step 5 are not needed if $y_s = 0$ implies $x_{sj} = 0$, e.g., if all the a_{sj} are nonnegative and the x_j are constrained to be nonnegative.

A somewhat similar approach to the disjunctive/scenario approach is the RLT approach developed by Adams and Sherali(2005). The next section illustrates the scenario approach for representing a decision problem.

11.2.7 Linearizing a Piecewise Linear Function, Discontinuous Case

If you ask a vendor to provide a quote for selling you some quantity of material, the vendor will typically offer a quantity discount function that looks something like that shown in Figure 11.2

Figure 11.2 Quantity Discount Piecewise Linear Discontinuous Cost Curve



Define:

c_s = slope of piecewise linear segment s,

h_s, v_s = horizontal and vertical coordinates of the rightmost point of segment s.

Note that segment 1 is the degenerate segment of buying nothing. This example illustrates that we do not require that a piecewise linear function be continuous.

Let us consider the following situation:

We pay \$50 if we buy anything in a period, plus
\$2.00/unit if quantity < 100,
\$1.90/unit if quantity ≥ 100 but < 1000,
\$1.80/unit if ≤1000 but ≤ 5000.

We assume h_s, v_s, c_s are constants, and $h_s \le h_{s+1}$. It then follows that:

h	v	c =
0	0	0
100	250	2
1000	1950	1.90
5000	9050	1.80;

We will describe two ways of representing picecewise linear functions: first the disjunctive method, and then the convex weighting or lambda method. Let x denote the amount we decide to purchase. Using step 1 of the scenario or disjunctive formulation approach,

if segment/scenario 1 is chosen, then
$cost = 0$;
$x = 0$;

If segment/scenario 2 is chosen, then
$cost = v_2 - c_2*(h_2 - x)$; [or $250 - 2*(100 - x)$],
$x \le h_2$; [or $x \le 100$],
$x \ge h_1$; [or $x \ge 0$],

Similar constraints apply for scenario/segments 3 and 4. We assume that fractional values, such as x = 99.44 are allowed, else we would write $x \le 99$ rather than $x \le 100$ above.

If we apply steps 2-4 of the scenario formulation method, then we get:

For segment/scenario 1 is chosen, then
$cost_1 = 0$;
$x_1 = 0$;

If segment/scenario 2 is chosen, then
$cost_2 = v_2*y_2 - c_2*h_2*y_2 + c_2*x_2$; [or $cost_2 = 50*y_2 + 2*x_2$],
$x_2 \le h_2*y_2$; [or $x_2 \le 100*y_2$],
$x_2 \ge h_1*y_2$; [or $x_2 \ge 0*y_2$],

If segment/scenario 3 is chosen, then
$cost_3 = v_3*y_3 - c_3*h_3*y_3 + c_3*x_3$; [or $cost_3 = 50*y_3 + 1.9*x_3$],
$x_3 \le h_3*y_3$; [or $x_3 \le 1000*y_3$],
$x_3 \ge h_2*y_3$; [or $x_3 \ge 100*y_3$],

If segment/scenario 4 is chosen, then

$cost_4 = v_4*y_4 - c_4*h_4*y_4 + c_4*x_4$; [or $cost_4 = 50*y_3 + 1.8*x_4$]
$x_4 \le h_4*y_4$; [or $x_3 \le 5000*y_4$],
$x_4 \ge h_3*y_4$; [or $x_3 \ge 1000*y_4$],

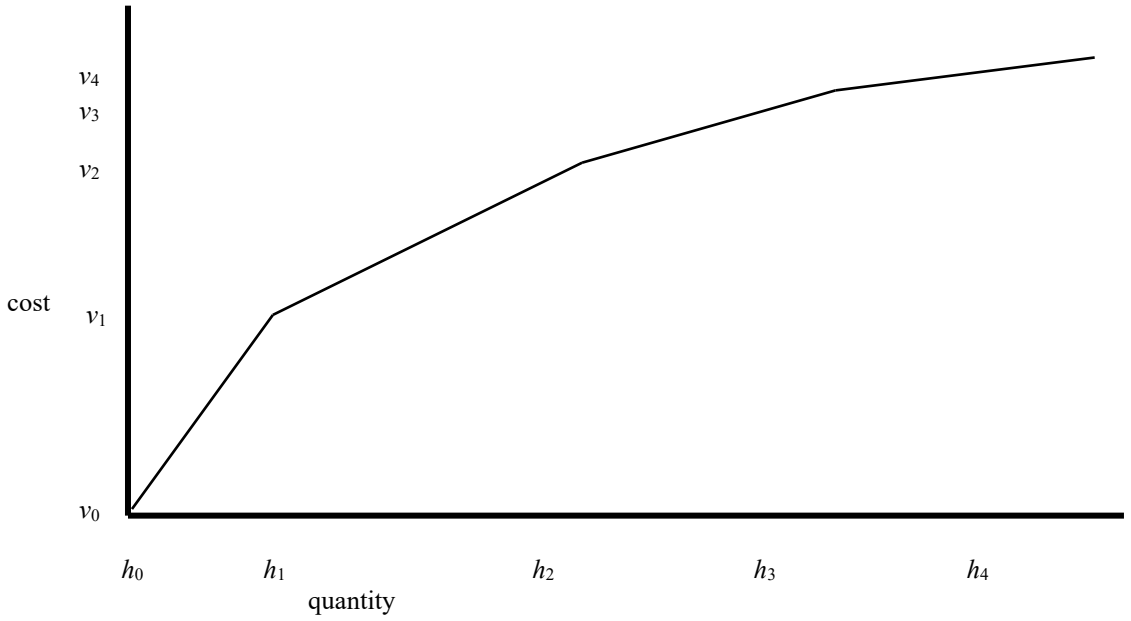
We must choose one of the four segments, so:
$y_1 + y_2 + y_3 + y_4 = 1$;
$y_1, y_2, y_3, y_4 = 0$ or 1;

and the true quantity and cost are found with:
$x_1 + x_2 + x_3 + x_4 = x$;
$cost_1 + cost_2 + cost_3 + cost_4 = cost$;

11.2.8 Linearizing a Piecewise Linear Function, Continuous Case

The previous quantity discount example illustrated what is called an "all units discount". Sometimes, a vendor will instead quote an incremental units discount, in which the discount applies only to the units above a threshold. The following example illustrates. The first 1,000 liters of the product can be purchased for $2 per liter. The price drops to $1.90 per liter for units beyond 1000, $1.80 for units above 3500, and $1.75 for units beyond 5000. At most 7000 liters can be purchased.

Figure 11.3 Continuous Piecewise Linear Cost Curve



Verify that the corresponding values for the h_i and v_i are:

i	h	v
0	0	0
1	1000	2000
2	3500	6750
3	5000	9450
4	7000	12950

Such continuous piecewise linear functions are found not only in purchasing but also are frequently used in the modeling of energy conversion processes such as the generation of electricity. The amount of electrical energy produced by a hydro-electric or fossil fuel burning generator may be a nonlinear function of the input volume of water or fuel.

Define the variables:

w_i = nonnegative weight to be applied to point i, for $i = 0, 1, 2, 3, 4$.
x = amount purchased,
cost = total cost of the purchase.

We can cause x and *cost* to almost be calculated correctly by writing the constraints:

$$x = w_0h_0 + w_1h_1 + w_2h_2 + w_3h_3 + w_4h_4;$$
$$\text{cost} = w_0v_0 + w_1v_1 + w_2v_2 + w_3v_3 + w_4v_4;$$
$$1 = w_0 + w_1 + w_2 + w_3 + w_4v_4;$$

Any point on the line segment connecting the two points (h_i, v_i) and (h_{i+1}, v_{i+1}) can be represented by choosing appropriate values for w_i and w_{i+1} so that $w_i + w_{i+1} = 1$, and $w_i, w_{i+1} \geq 0$. This method is sometimes called the lambda method because the Greek symbol lambda was used originally to represent the weights. To ensure that the point corresponding to a particular set of values for the w_i lies on the curve, we need to require that if two or more of the w_i are > 0, they must be adjacent. We said "almost" in the earlier sentence because there is nothing in the three constraints above that enforce this adjacency condition. There are two ways of enforcing this adjacency condition: a) declare the w_i to be members of an SOS2 set in LINGO, or b) add a number of binary variables to enforce the condition.

The following code fragment illustrates how to use the SOS2 feature in LINGO.

```
! Representing a continuous piecewise linear
   function in LINGO using the SOS2 feature;

         x = w0*0 + w1*1000 + w2*3500 + w3*5000 + w4* 7000;
      cost = w0*0 + w1*2000 + w2*6750 + w3*9450 + w4*12950;
         1 = w0   + w1      + w2      + w3      + w4;
 ! The ordering/adjacency of the variables in the SOS2 set
   is determined by the order of declarations. The SOS2 feature
   restricts the number of nonzero values in the set to be at
   most 2, and if 2, they must be adjacent;
    @SOS2('MySOS2',w0); @SOS2('MySOS2',w1); @SOS2('MySOS2',w2);
    @SOS2('MySOS2',w3); @SOS2('MySOS2',w4);
```

If you arbitrarily add the constraint, X = 6000, and solve, you get the solution:

```
                Variable           Value
                       X        6000.000
                      W0        0.000000
                      W1        0.000000
                      W2        0.000000
                      W3       0.5000000
                      W4       0.5000000
                    COST        11200.00
```

If for some reason you do not want to use the SOS2 feature, you can introduce 4 binary variables:

$y_i = 1$ if x is in the interval with endpoints h_{i-1} and h_i, for $i = 1, 2, 3, 4$. We would replace the SOS2 declarations by the constraints:

```
! The y's must be binary;
    @BIN(y1); @BIN(y2); @BIN(y3); @BIN(y4);
! Some interval must be chosen;
            y1 + y2 + y3 + y4 = 1;
! If point i has any weight, then one of the adjacent
   intervals must be chosen;
        w0 <= y1;
        w1 <= y1 + y2;
        w2 <= y2 + y3;
        w3 <= y3 + y4;
        w4 <= y4;
```

11.2.9 An *n* Interval Piecewise Linear Function Using Log(*n*) Binaries

The previous example used n binary variables to enforce the choosing of one alternative out of n. With a little ingenuity this "choose one out of n" requirement can be enforced with only order of $\log_2(n)$ binary variables. We illustrate for the case of eight intervals, for which we need three binary variables, y_1, y_2, y_3. Denote the 8 intervals by 0, 1, …,7, with point v_i being the left boundary of interval i. We will assign binary variables to intervals thus:

If the interval is one of 4, 5, 6, 7, then $y_3 = 1$, else 0,
If the interval is one of 2, 3, 4, 5, then $y_2 = 1$, else 0,
If the interval is one of 1, 2, 5, 6, then $y_1 = 1$, else 0;

Thus, we also need the constraints:

$$w_0+w_1+w_2+w_3 \le 1- y_3;$$
$$w_5+w_6+w_7+w_8 \le y_3;$$
$$w_0+w_1+w_7+w_8 \le 1- y_2;$$
$$w_3+w_4+w_5 \le y_2;$$
$$w_0+w_4+w_8 \le 1- y_1;$$
$$w_2+ w_6 \le y_1;$$
$$y_1, y_2, y_3 = 0 \text{ or } 1;$$

Notice that if:

$y_1 = y_2 = y_3 = 0$, then $w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 = 0$, i.e., the interval is 0,

$y_1 = 1, y_2 = y_3 = 0$, then $w_0 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 = 0$, i.e., the interval is 1, etc.

It may be of interest to note that this is a "Gray" binary coding of 0, 1, …,7, in that exactly 1 "bit" of y_1, y_2, y_3 changes in the binary representation as one moves from i to i+1.

Piecewise Linear Approximations to Multivariate Functions

Suppose we have a function of two variables:

$cost = f(x,y)$.

We can construct a piecewise linear approximation to this function if we

choose n points, $(xbar_i, ybar_i)$ for $i = 1, 2, \ldots, n$,

e.g., corner points of the triangles in Figure 11.4, and,

introduce the n nonnegative variables, w_i, and

add the constraints:

$\Sigma_i w_i = 1$,

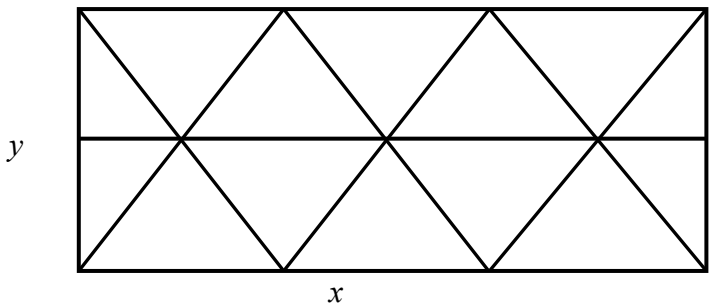
$cost = \Sigma_i w_i f(xbar_i, ybar_i)$,

$x = \Sigma_i w_i xbar_i$,

$y = \Sigma_i w_i ybar_i$,

If we are lucky, e.g., $f(x,y)$ is convex in the appropriate way then: a) at most three of the w_i will be nonzero, and b) the nonzero w_i will correspond to adjacent points, that is, corner points of a triangle containing no other points.

Figure 11.4 Triangulation of x,y Space



If we are unlucky, then we have to introduce 0/1 variables. If we are lazy and are willing to restrict the solution to one of the n sampled points, then all we have to do is declare the w_i variables to be 0/1.

If we want to allow any possible combination of x and y, then we have to make sure the n points describe a triangulation of the x,y space, as in Figure 11.4, introduce a 0/1 variable z_j for each triangle, and then force exactly one triangle to be chosen with the constraint:

$\Sigma_i z_i = 1$;

We must also add constraints that say that if any weight is applied to point i, then the chosen (x,y) must be in one of the triangles for which point i is a corner. More formally:

$w_i \le \Sigma_{j \text{ in } T(i)} z_j$, for each point i, where $T(i)$ is the

set of triangles(there should be at most 6) for which point i is a corner point.

11.2.10 Converting Multivariate Functions to Separable Functions

The previous methods are applicable only to piecewise linear functions. There are some standard methods available for transforming certain functions of several variables, so a function is obtained that is additively separable in the transformed variables. The most common such transformation is for converting a product of two variables into separable form. For example, given the function:

$$x_1 * x_2 ,$$

add the linear constraints:

$$y_1 = (x_1 + x_2)/2$$
$$y_2 = (x_1 - x_2)/2.$$

Then, replace every instance of $x_1 * x_2$ by the term $y_1^2 - y_2^2$. That is, the claim is:

$$x_1 * x_2 = y_1^2 - y_2^2.$$

The justification is observed by noting:

$$y_1^2 - y_2^2 = (x_1^2 + 2 * x_1 * x_2 + x_2^2)/4$$
$$- (x_1^2 - 2 * x_1 * x_2 + x_2^2)/4$$
$$= 4 * x_1 * x_2 /4 = x_1 * x_2$$

This example suggests that, any time you have a product of two variables, you can add two new variables to the model and replace the product term by a sum of two squared variables. If you have n original variables, you could have up to $n(n-1)/2$ cross product terms. This suggests that you might need up to $n(n-1)$ new variables to get rid of all cross product terms. In fact, the above ideas can be generalized, using various factorization techniques such as Cholesky and others, so only n new variables are needed.

11.3 Outline of Integer Programming Methods

The time a computer requires to solve an IP may depend dramatically on how you formulated it. It is, therefore, worthwhile to know a little about how IPs are solved. There are two general approaches for solving IPs: "cutting plane" methods and "branch-and-bound" (B & B) method. For a comprehensive introduction to integer programming solution methods, see Nemhauser and Wolsey (1988), and Wolsey (1998). Most commercial IP programs use the B & B method, but aided by some cutting plane features. We will first describe the B & B method. In most general terms, B & B is a form of intelligent enumeration.

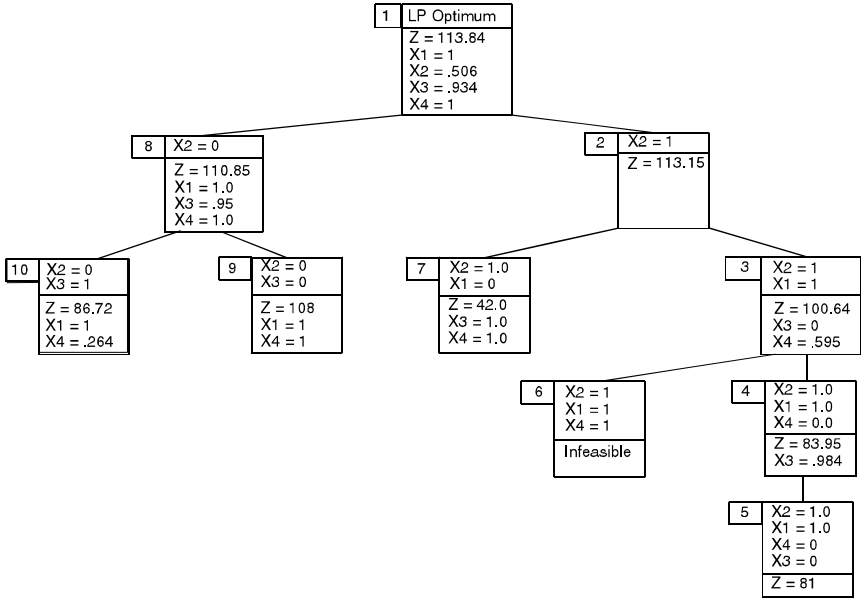
More specifically, B & B first solves the problem as an LP. If the LP solution is integer valued in the integer variables, then no more work is required. Otherwise, B & B resorts to an intelligent search of all possible ways of rounding the fractional variables.

We shall illustrate the application of the branch-and-bound method with the following problem:

```
MAX= 75 * X1  + 6 * X2   + 3 * X3 + 33 * X4;
     774 * X1 + 76 * X2  + 22 * X3 + 42 * X4 <= 875;
      67 * X1 + 27 * X2 + 794 * X3 + 53 * X4 <= 875;
     @BIN( X1); @BIN( X2); @BIN( X3); @BIN( X4);
```

The search process a computer might follow in finding an integer optimum is illustrated in Figure 11.5. First, the problem is solved as an LP with the constraints $X1, X2, X3, X4 \leq 1$. This solution is summarized in the box labeled 1. The solution has fractional values for *X2* and *X3* and is, therefore, unacceptable. At this point, *X2* is arbitrarily selected and the following reasoning is applied. At the integer optimum, *X2* must equal either 0 or 1.

Figure 11.5 Branch-and-Bound Search Tree



Therefore, replace the original problem by two new subproblems. One with *X2* constrained to equal 1 (box or node 2) and the other with *X2* constrained to equal 0 (node 8). If we solve both of these new IPs, then the better solution must be the best solution to the original problem. This reasoning is the motivation for using the term "branch". Each subproblem created corresponds to a branch in an enumeration tree.

The numbers to the upper left of each node indicate the order in which the nodes (or equivalently, subproblems) are examined. The variable *Z* is the objective function value. When the subproblem with *X2* constrained to 1 (node 2) is solved as an LP, we find *X1* and *X3* take fractional values. If we argue as before, but now with variable *X1*, two new subproblems are created:

Node 7) one with *X1* constrained to 0 , and
Node 3) one with *X1* constrained to 1.

This process is repeated with *X4* and *X3* until node 5. At this point, an integer solution with $Z = 81$ is found. We do not know this is the optimum integer solution, however, because we must still look at subproblems 6 through 10. Subproblem 6 need not be pursued further because there are no feasible solutions having all of *X2*, *X1*, and *X4* equal to 1. Subproblem 7 need not be pursued further because it has a *Z* of 42, which is worse than an integer solution already in hand.

At node 9, a new and better integer solution with $Z = 108$ is found when *X3* is set to 0. Node 10 illustrates the source for the "bound" part of "branch-and-bound". The solution is fractional. However, it is not examined further because the *Z*-value of 86.72 is less than the 108 associated with an integer solution already in hand. The *Z*-value at any node is a bound on the *Z*-value at any offspring node. This is true because an offspring node or subproblem is obtained by appending a constraint to the parent problem. Appending a constraint can only hurt. Interpreted in another light, this means the *Z*-values cannot improve as one moves down the tree. The tree presented in the preceding figure was only one illustration of how the tree might be searched. Other trees could be developed for the same problem by playing with the following two degrees of freedom:

(a) Choice of next node to examine, and
(b) Choice of how the chosen node is split into two or more subnodes.

For example, if nodes 8 and then 9 were examined immediately after node 1, then the solution with $Z = 108$ would have been found quickly. Further, nodes 4, 5, and 6 could then have been skipped because the *Z*-value at node 3 (100.64) is worse than a known integer solution (108), and, therefore, no offspring of node 3 would need examination.

In the example tree, the first node is split by branching on the possible values for *X2*. One could have just as well chosen *X3* or even *X1* as the first branching variable.

The efficiency of the search is closely related to how wisely the choices are made in (a) and (b) above. Typically, in (b) the split is made by branching on a single variable. For example, if, in the continuous solution, $x = 1.6$, then the obvious split is to create two subproblems. One with the constraint $x \le 1$, and the other with the constraint $x \ge 2$. The split need not be made on a single variable. It could be based on an arbitrary constraint. For example, the first subproblem might be based on the constraint $x_1 + x_2 + x_3 \le 0$, while the second is obtained by appending the constraint $x_1 + x_2 + x_3 \ge 1$. Also, the split need not be binary. For example, if the model contains the constraint $y_1 + y_2 + y_3 = 1$, then one could create three subproblems corresponding to either $y_1 = 1$, or $y_2 = 1$, or $y_3 = 1$.

If the split is based on a single variable, then one wants to choose variables that are "decisive." In general, the computer will make intelligent choices and the user need not be aware of the details of the search process. The user should, however, keep the general B & B process in mind when formulating a model. If the user has *a priori* knowledge that an integer variable *x* is decisive, then for the LINGO program it is useful to place *x* early in the formulation to indicate its importance. This general understanding should drive home the importance of a "tight" LP formulation. A tight LP formulation is one which, when solved, has an objective function value close to the IP optimum. The LP solutions at the subproblems are used as bounds to curtail the search. If the bounds are poor, many early nodes in the tree may be explicitly examined because their bounds look good even though, in fact, these nodes have no good offspring.

Cutting planes are very important for solving certain classes of IP's. Some of these difficult IP's would take prohibitively long to solve with just B&B, without the use of cutting planes. A cutting plane is an additional constraint that is added to the formulation to remove fractional points from the LP relaxation. Thus, if some good cuts have been added, the chance is much higher that when the LP relaxation is solved, a much higher fraction of the integer variables will take on naturally integer values.

There is a wide variety of cuts that are implemented in commercial IP solvers, an even wider variety of cuts that have been described in the optimization literature. One of the most general types of cuts is the Mixed Integer Rounding, or MIR, cut. A very similar cut described in the literature is the Gomory Mixed Integer cut. We will illustrate the MIR cut with a little example. Suppose we want to solve the little mixed integer program:

```
MIN = 5*y + 3*u + 4*v;
```

```
        8*y + u - v = 13;
    @GIN( y)
```

As usual, by default, all variables are restricted to be ≥ 0. If you delete the requirement that y be integer and solve the resulting LP, you get the fractional solution $y = 1.625$, $u = v = 0$. Now we reason that in any feasible integer solution, either :

Case 1: $y \leq 1$ and $u \geq 5$, or

Case 2: $y \geq 2$ and $v \geq 3$.

So, in any integer feasible solution, we must have either:

$u \geq 5$, or

$v \geq 3$.

Multiplying by either 3 or 5, we must have either:

$3*u \geq 3*5$, or

$5*v \geq 3*5$.

Because v, $u \geq 0$, we can add $5*v$ to the first constraint, and $3*u$ to the second constraint without destroying their validity, so we must have either:

$3*u + 5*v \geq 3*5$, or

$3*u + 5*v \geq 3*5$,

so the single constraint or cut is justified:

$3*u + 5*v \geq 3*5$.

Notice that this cut cuts off the fractional solution $y = 1.625$, $u = v = 0$. With this cut added, when we solve the LP:

```
MIN = 5*y + 3*u + 4*v;
      8*y + u - v = 13;
      3*u + 5*v ≥ 15;
```

We get the naturally integer solution;

```
Global optimal solution found.
  Objective value:              20.000000

             Variable           Value
                    Y        1.000000
                    U        5.000000
                    V        0.000000
```

With a little bit of imagination, e.g., by replacing y by a sum of integer variables with integer coefficients, and replacing u and v by positive weighted sums of nonnegative variables, a wide variety of MIR type cuts are possible.

11.4 Computational Difficulty of Integer Programs

Integer programs can be very difficult to solve. This is in marked contrast to LP problems. The solution time for an LP is fairly predictable. For an LP, the time increases approximately proportionally with the number of variables and approximately with the square of the number of constraints. For a given IP problem, the time may in fact decrease as the number of constraints is increased. As the number of integer variables is increased, the solution time may increase dramatically. Some small IPs (e.g., 6 constraints, 60 variables) are extremely difficult to solve.

Just as with LPs, there may be alternate IP formulations of a given problem. With IPs, however, the solution time generally depends critically upon the formulation. Producing a good IP formulation requires skill. For many of the problems in the remainder of this chapter, the difference between a good formulation and a poor formulation may be the difference between whether the problem is solvable or not.

11.4.1 NP-Complete Problems

Integer programs belong to a class of problems known as *NP*-hard. We may somewhat loosely think of *NP* as meaning "not polynomial". This means that there is no known algorithm of solving these problems such that the computational effort at worst increases as a polynomial in the problem size. For our purposes, we will say that the computational complexity of an algorithm is polynomial if there is a positive constant k, such that the time to solve a problem of size n is proportional to n^k. For example, sorting a set of n numbers can easily be done in (polynomial) time proportional to n^2,($n\ log(n)$ if one is careful), whereas solving an integer program in n zero/one variables may, in the worst case, take (exponential) time proportional to 2^n. There may be a faster way, but no one has published an algorithm for integer programs that is guaranteed to take polynomial time on every problem presented to it. The terms *NP*-complete and *P*-complete apply to problems that can be stated as "yes/no" or feasibility problems. The yes/no variation of an optimization problem would be a problem of the form: Is there a feasible solution to this problem with cost less-than-or-equal-to 1250. In an optimization problem, we want a feasible solution with minimum cost. Khachian (1979) showed that the feasibility version of LP is solvable in polynomial time. So, we say LP is in *P*. Integer programming stated in feasibility form, and a wide range of similar problems, belong to a class of problems called *NP*-complete. These problems have the feature that it is possible to convert any one of these problems into any other *NP*-complete problem in time that is polynomial in the problem size. Thus, if we can convert problem *A* into problem *B* in polynomial time, then solve *B* in polynomial time, and then convert the solution to *B* to a valid solution to *A* in polynomial time, we then have a way of solving *A* in polynomial time.

The notable thing about *NP*-complete problems is that, if someone develops a guaranteed fast (e.g., polynomial worst case) time method for solving one of these problems, then that someone also has a polynomial time algorithm for every other *NP*-complete problem. An important point to remember is that the *NP*-completeness classification is defined in terms of worst-case behavior, not average case behavior. For practical purposes, one is interested mainly in average case behavior. The current situation is that the average time to solve many important practical integer programming problems is quite short. The fact that someone may occasionally present us with an extremely difficult integer programming problem does not prevent us from profiting from the fact that a large number of practical integer programs can be solved rapidly. Perhaps the biggest open problem in modern mathematics is whether the problems in the *NP*-complete class are inherently difficult. This question is cryptically phrased as is: $P = NP$? Are these problems really difficult, or is it that we are just not smart enough to discover the universally fast algorithm? In fact, a “Millenium prize” of $1,000,000 is offered by the Clay Mathematics Institute, www.claymath.org, for an answer to this question. For a more comprehensive discussion of the *NP*-complete classification, see Martin (1999).

11.5 Problems with Naturally Integer Solutions and the Prayer Algorithm

The solution algorithms for IP are generally based on first solving the IP as an LP by disregarding the integrality requirements and praying the solution is naturally integer. For example, if x is required to be 0 or 1, the problem is first solved by replacing this requirement by the requirement that simply $0 \le x \le 1$.

When initiating the analysis of a problem in which integer answers are important, it is useful to know beforehand whether the resulting IP will be easy to solve. After the fact, one generally observes the IP was easy to solve if the objective function values for the LP optimum and the IP optimum were close. About the only way we can predict beforehand the objective function values of the LP and IP will be close is if we know beforehand the LP solution will be almost completely integer valued. Thus, we are interested in knowing what kinds of LPs have naturally integer solutions.

The classes of LP problems for which we know beforehand there is a naturally integer optimum have integer right-hand sides and are in one of the classes:

(a) Network LPs,
(b) MRP or Integral Leontief LPs,
(c) Problems that can be transformed to (a) or (b) by either row operations or taking the dual.

We first review the distinguishing features of network and MRP LPs.

11.5.1 Network LPs Revisited

A LP is said to be a network LP if: 1) disregarding simple upper and lower bound constraints (such as $x \leq 3$), each variable appears in at most two constraints, and 2) if each variable appears in two constraints, its coefficients in the two are +1 and -1. If the variable appears in one constraint, its coefficient is either +1 or -1.

Result: If the right-hand side is integer, then there is an integer optimum. If the objective coefficients are all integer, then there is an optimum with integral dual prices.

11.5.2 Integral Leontief Constraints

A constraint set is said to be *integral Leontief* or MRP (for Material Requirements Planning) if (see Jeroslow, Martin, et al. (1992)):

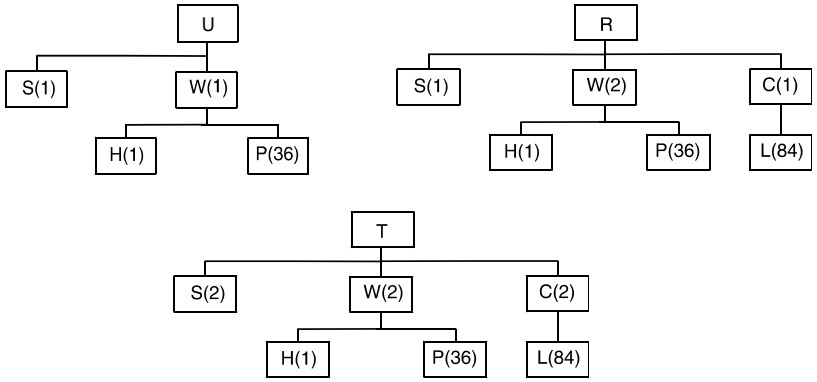
- Each constraint is an equality,
- Every column has exactly one positive coefficient and it is a +1,
- Each column has 0 or more negative coefficients, every one of which is integer,
- Each RHS coefficient is a nonnegative integer.

Result: An LP whose complete constraint set is an MRP set has an optimal solution that is integer. Further, if the objective coefficients are all integer, then there is an optimal solution with integral dual prices.

11.5.3 Example: A One-Period MRP Problem

The Schwindle Cycle Company makes three products: Unicycles (*U*), Regular Bicycles (*R*), and Twinbikes (*T*). Each product is assembled from a variety of components including: seats (*S*), wheels (*W*), hubs (*H*), spokes (*P*), chains (*C*), and links (*L*). The full bills of materials for each product are shown below. The numbers in parentheses specify how many units of the child are required per parent:

Figure 11.6 MRP Structure for Bicycles



Current inventories are zero. Schwindle needs to supply 100 Unicycles, 500 Regular bicycles, and 200 Twinbikes. Finished products and complete sub-assemblies can be either manufactured or bought at the following prices:

Item:	**U**	**R**	**T**	**S**	**W**	**C**	**H**	**P**	**L**
Bought Price:	2.60	5.2	3.10	0.25	1.40	0.96	0.19	0.07	0.05
Assembly Cost:	1.04	1.16	1.90	0.20	0.22	0.26	0.16	0.04	0.03

Note the assembly cost is the immediate cost at the level of assembly. It does not include the cost of the components going into the assembly. How many units of each item should be made or bought to satisfy demand at minimum price?

An LP formulation is:

```
MODEL:
  SETS:
    TYPES/U, R, T/:M, B, MP, BP, NEED;
    MATERIALS/S, W, C/:MM, MB, MMP, MBP;
    SUBMATS/H, P, L/:SMM, SMB, SMP, SBP;
    REQ(TYPES, MATERIALS): MATREQ;
    MREQ(MATERIALS, SUBMATS): SMATREQ;
  ENDSETS
  DATA:
    NEED    =  100  500  200;
    MP      = 1.04 1.16  1.9;
    BP      =  2.6  5.2  3.1;
    MMP     =   .2  .22  .26;
    MBP     =  .25  1.4  .96;
    SMP     =  .16  .04  .03;
    SBP     =  .19  .07  .05;
    MATREQ  =    1    1    0
                 1    2    1
                 2    2    2;
    SMATREQ =    0    0    0
                 1   36    0
                 0    0   84;
  ENDDATA
    MIN = @SUM(TYPES : M * MP + B * BP)
        + @SUM(MATERIALS : MM * MMP + MB * MBP)
        + @SUM(SUBMATS: SMM * SMP + SMB * SBP);
    @FOR(TYPES: M + B = NEED);
    @FOR(MATERIALS(I): MM(I) + MB(I) =
        @SUM(TYPES(J): M(J) * MATREQ(J, I)));
    @FOR(SUBMATS(I): SMM(I) + SMB(I) =
        @SUM(MATERIALS(J): MM(J) * SMATREQ(J, I)));
END
```

In the PICTURE of the formulation below, notice it has the MRP structure:

```
           U U R R T T S S W W C C H H P P L L
           M B M B M B M B M B M B M B M B M B
        1: A A A A A A T T T A T T T T U U U U MIN
UNICYCLE: 1 1   '     '     '     '     '     = B
 REGULAR: '  '1 1  '  '  '  '  '  '  '  '  '  = C
TWINBIKE:       ' 1 1 '     '                 = C
   SEATS:-1  -1 '-2   1 1   '     '     '     =
  WHEELS:-1  -2 '-2'  '  '1 1  '  '  '  '  '  =
  CHAINS:    -1 '-2   '     ' 1 1 '     '     =
    HUBS:       '     '  -1 '     1 1   '     =
  SPOKES: '  '  '  '  '  -B '  '  '  '1 1  '  =
   LINKS:       '     '     '-B   '     ' 1 1 =
```

The solution is:

```
Optimal solution found at step:          0
Objective value:                  3440.000
Variable           Value        Reduced Cost
   M( R)         500.0000           0.0000000
   B( U)         100.0000           0.0000000
   B( T)         200.0000           0.0000000
  MM( S)         500.0000           0.0000000
  MB( W)         1000.000           0.0000000
  MB( C)         500.0000           0.0000000
```

Notice it is naturally integer. Thus, we should buy all the unicycles and twin bikes (and paste our own brand name on them). We assemble our own regular bicycles. They are assembled from manufactured seats and bought wheels and chains.

If we put an upper limit of 300 on the number of links manufactured by adding the constraint $LM \leq 300$, we will get a fractional solution because this constraint violates the MRP structure.

11.5.4 Transformations to Naturally Integer Formulations

A *row operation* consists of either of the following:

- multiplication through an equation by some non-zero constant,
- adding a finite multiple of one equation to another.

A row operation changes neither the feasible region nor the set of optimal solutions to a problem. Thus, if we can show a model can be transformed to either a network LP or an MRP LP by row operations, then we know there is an integer optimum. We do not actually need to do the transformation to get the solution.

Similarly, if we have a model with an integer right-hand side and we can show it is the dual of either a network LP or an MRP LP, then we know the model has an integer optimum.

Example

Consider the following LP that arose in planning how much to produce in each of four periods:

```
     P P P P P P P P P P
     1 1 1 1 2 2 2 3 3 4
     4 3 2 1 4 3 2 4 3 4
1:   9 6 4 3 6 4 3 4 3 3 MIN
2:   1 1 1 1                 = 1
3:   1 1 1     1 1 1         = 1
4:   1 1       1 1     1 1   = 1
5:   1         1       1   1 = 1
```

When solved as an LP, we obtained the following naturally integer solution:

$P12 = P34 = 1$; all others 0.

Could we have predicted a naturally integer solution beforehand? If we perform the row operations: (5') = (5) – (4); (4') = (4) – (3); (3') = (3) – (2), we obtain the equivalent LP:

```
     P P P P P P P P P P
     1 1 1 1 2 2 2 3 3 4
     4 3 2 1 4 3 2 4 3 4
 1: 9 6 4 3 6 4 3 4 3 3 MIN
 2: 1 1 1 1             = 1
3':       -1 1 1 1     = 0
4':     -1       -1 1 1 = 0
5':   -1       -1     -1 1 = 0
```

This is a network LP, so it has a naturally integer solution.

Example
In trying to find the minimum elapsed time for a certain project composed of seven activities, the following LP was constructed (in PICTURE form):

```
    A B C D E F
 1:-1      '   1 MIN
AB:-1 1    '     >= 3
AC:-1   '1 '  '  >= 2
BD:  -1    1     >= 5
BE:  -1    ' 1   >= 6
CF: '  -1 '   '1 >= 4
DF:       -1   1 >= 7
EF:        '-1 1 >= 6
```

This is neither a network LP (e.g., consider columns *A*, *B*, or *F*) nor an MRP LP (e.g., consider columns *A* or *F*). Nevertheless, when solved, we get the naturally integer solution:

```
Optimal solution found at step:          0
Objective value:                  15.00000

Variable            Value        Reduced Cost
       A        0.0000000           0.0000000
       B         3.000000           0.0000000
       C         2.000000           0.0000000
       D         8.000000           0.0000000
       E         9.000000           0.0000000
       F         15.00000           0.0000000

     Row    Slack or Surplus      Dual Price
       1         15.00000           1.000000
      AB        0.0000000          -1.000000
      AC        0.0000000           0.0000000
      BD        0.0000000          -1.000000
      BE        0.0000000           0.0000000
      CF         9.000000           0.0000000
      DF        0.0000000          -1.000000
      EF        0.0000000           0.0000000
```

Could we have predicted a naturally integer solution beforehand? If we look at the PICTURE of the model, we see each constraint has exactly one +1 and one −1. Thus, its dual model is a network LP and expectation of integer answers is justified.

11.6 The Assignment Problem and Related Sequencing and Routing Problems

The assignment problem is a simple LP problem, which is frequently encountered as a major component in more complicated practical problems.

The assignment problem is:

Given a matrix of costs:

c_{ij} = cost of assigning object i to person j,

and variables:

$x_{ij} = 1$ if object i is assigned to person j.

Then, we want to:

Minimize $\sum_i \sum_j c_{ij} x_{ij}$

subject to

$\sum_i x_{ij} = 1$ for each object i,

$\sum_j x_{ij} = 1$ for each person i,

$x_{ij} \geq 0$.

This problem is easy to solve as an LP and the x_{ij} will be naturally integer.

There are a number of problems in routing and sequencing that are closely related to the assignment problem.

11.6.1 Example: The Assignment Problem

Large airlines tend to base their route structure around the hub concept. An airline will try to have a large number of flights arrive at the hub airport during a certain short interval of time (e.g., 9 A.M. to 10 A.M.) and then have a large number of flights depart the hub shortly thereafter (e.g., 10 A.M. to 11 A.M.). This allows customers of that airline to travel between a large combination of origin/destination cities with one stop and at most one change of planes. For example, United Airlines uses Chicago as a hub, Delta Airlines uses Atlanta, and American uses Dallas/Fort Worth.

A desirable goal in using a hub structure is to minimize the amount of changing of planes (and the resulting moving of baggage) at the hub. The following little example illustrates how the assignment model applies to this problem.

A certain airline has six flights arriving at O'Hare airport between 9:00 and 9:30 A.M. The same six airplanes depart on different flights between 9:40 and 10:20 A.M. The average numbers of people transferring between incoming and leaving flights appear below:

	L01	**L02**	**L03**	**L04**	**L05**	**L06**
I01	20	15	16	5	4	7
I02	17	15	33	12	8	6
I03	9	12	18	16	30	13
I04	12	8	11	27	19	14
I05	0	7	10	21	10	32
I06	0	0	0	6	11	13

Flight I05 arrives too late to connect with L01. Similarly I06 is too late for flights L01, L02, and L03.

All the planes are identical. A decision problem is which incoming flight should be assigned to which outgoing flight. For example, if incoming flight *I02* is assigned to leaving flight *L03*, then 33 people (and their baggage) will be able to remain on their plane at the stop at O'Hare. How should incoming flights be assigned to leaving flights, so a minimum number of people need to change planes at the O'Hare stop?

This problem can be formulated as an assignment problem if we define:

x_{ij} = 1 if incoming flight i is assigned to outgoing flight j,
0 otherwise.

The objective is to maximize the number of people not having to change planes. A formulation is:

```
MODEL:   !  Assignment model(ASSIGNMX);
SETS:
 FLIGHT;
 ASSIGN( FLIGHT, FLIGHT): X, CHANGE;
ENDSETS
DATA:
 FLIGHT = 1..6;
! The value of assigning i to j;
 CHANGE = 20   15   16   5   4   7
          17   15   33  12   8   6
           9   12   18  16  30  13
          12    8   11  27  19  14
        -999    7   10  21  10  32
        -999 -999 -999   6  11  13;
ENDDATA
!---------------------------------;
! Maximize value of assignments;
MAX = @SUM(ASSIGN: X * CHANGE);
@FOR( FLIGHT( I):
!  Each I must be assigned to some J;
    @SUM( FLIGHT( J): X( I, J)) = 1;
!  Each I must receive an assignment;
    @SUM( FLIGHT( J): X( J, I)) = 1;
     );
END
```

Notice, we have made the connections that are impossible prohibitively unattractive. A solution is:

```
Optimal solution found at step:          9
Objective value:                   135.0000

Variable           Value        Reduced Cost
X( 1, 1)        1.000000           0.0000000
X( 2, 3)        1.000000           0.0000000
X( 3, 2)        1.000000           0.0000000
X( 4, 4)        1.000000           0.0000000
X( 5, 6)        1.000000           0.0000000
X( 6, 5)        1.000000           0.0000000
```

Notice, each incoming flight except *I03* is able to be assigned to its most attractive outgoing flight. The solution is naturally integer even though we did not declare any of the variables to be integer.

11.6.2 The Traveling Salesperson Problem

One of the more famous optimization problems is the traveling salesperson problem (TSP). It is an assignment problem with the additional condition that the assignments chosen must constitute a tour. The objective is to minimize the total distance traveled. Lawler et al. (1985) presents a tour-de-force on this fascinating problem. One example of a TSP occurs in the manufacture of electronic circuit boards. Danusaputro, Lee, and Martin-Vega (1990) discuss the problem of how to optimally sequence the drilling of holes in a circuit board, so the total time spent moving the drill head between holes is minimized. A similar TSP occurs in circuit board manufacturing in determining the sequence in which components should be inserted onto the board by an automatic insertion machine. Another example is the sequencing of cars on a production line for painting: each time there is a change in color, a setup cost and time is incurred.

A TSP is described by the data:

c_{ij} = cost of traveling directly from city i to city j, e.g., the distance.

A solution is described by the variables:

$y_{ij} = 1$ if we travel directly from i to j, else 0.

The objective is:

Min $\Sigma_i \Sigma_j \, c_{ij} \, y_{ij}$;

We will describe several different ways of specifying the constraints.

Subtour Elimination Formulation:

(1) We must enter each city j exactly once:

$$\sum_{i \neq j}^{n} y_{ij} = 1 \qquad \text{for } j = 1 \text{ to } n,$$

(2) We must exit each city i exactly once:

$$\sum_{j \neq i}^{n} y_{ij} = 1 \qquad \text{for } i = 1 \text{ to } n,$$

(3) $y_{ij} = 0$ or 1, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i \neq j$:

(4) No subtours are allowed for any subset of cities S not including city 1:

$$\sum_{i,j \in S} y_{ij} \leq |S| - 1 \qquad \text{for every subset } S,$$

where $|S|$ is the size of S.

The above formulation is usually attributed to Dantzig, Fulkerson, and Johnson(1954). An unattractive feature of the Subtour Elimination formulation is that if there are n cities, then there are approximately 2^n constraints.

Cumulative Load Formulation:

We can reduce the number of constraints substantially if we define: u_j = the sequence number of city j on the trip. Equivalently, if each city requires one unit of something to be picked up(or delivered), then u_j = cumulative number of units picked up(or delivered) after the stop at j. We replace constraint set (4) by:

(5) $u_j \geq u_i + 1 - (1 - y_{ij})n$ for $i = 1, 2, ..., j = 2, 3, 4, \ldots ; j \neq i$.

The approach of constraint set (5) is due to Miller, Tucker, and Zemlin(1960). There are only approximately n^2 constraints of type (5), however, constraint set (4) is much tighter than (5). Large problems may be computationally intractable if (4) is not used. Even though there are a huge number of constraints in (4), only a few of them may be binding at the optimum. Thus, an iterative approach that adds violated constraints of type (4) as needed works surprisingly well. Padberg and Rinaldi (1987) used essentially this iterative approach and were able to solve to optimality problems with over 2000 cities. The solution time was several hours on a large computer.

Multi-commodity Flow Formulation:

Similar to the previous formulation, imagine that each city needs one unit of some commodity distinct to that city. Define:

x_{ijk} = units of commodity carried from i to j, destined for ultimate delivery to k.

If we assume that we start at city 1 and there are n cities, then we replace constraint set (4) by:

For $k = 1, 2, 3, \ldots, n$:

$\Sigma_{j>1} x_{1jk} = 1$; (Each unit must be shipped out of the origin.)

$\Sigma_{i \neq k} x_{ikk} = 1$; (Each city k must get its unit.)

For $j = 2, 3, \ldots, n$, $k = 1, 2, 3, \ldots, n$, $j \neq k$:

$\Sigma_i x_{ijk} = \Sigma_{t \neq j} x_{jtk}$ (Units entering j, but not destined for j, must depart j to some city t.)

A unit cannot return to 1, except if its final destination is 1:

$\Sigma_i \Sigma_{k>1} x_{i1k} = 0$,

For $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n$, $i \neq j$:

$x_{ijk} \leq y_{ij}$ (If anything shipped from i to j, then turn on y_{ij}.)

The drawback of this formulation is that it has approximately n^3 constraints and variables. A remarkable feature of the multicommodity flow formulation is that it is just as tight as the Subtour Elimination formulation. The multi-commodity formulation is due to Claus(1984).

Heuristics

For practical problems, it may be important to get good, but not necessarily optimal, answers in just a few seconds or minutes rather than hours. The most commonly used heuristic for the TSP is due to Lin and Kernighan (1973). This heuristic tries to improve a given solution by clever re-orderings of cities in the tour. For practical problems (e.g., in operation sequencing on computer controlled machines), the heuristic seems always to find solutions no more than 2% more costly than the optimum. Bland and Shallcross (1989) describe problems with up to 14,464 "cities" arising from the sequencing of operations on a computer-controlled machine. In no case was the Lin-Kernighan heuristic more than 1.7% from the optimal for these problems.

Example of a Traveling Salesperson Problem

P. Rose, currently unemployed, has hit upon the following scheme for making some money. He will guide a group of 18 people on a tour of all the baseball parks in the National League. He is betting his life savings on this scheme, so he wants to keep the cost of the tour as low as possible. The tour will start and end in Cincinnati. The following distance matrix has been constructed:

	Atl	Chi	Cin	Hou	Lax	Mon	NYk	Phi	Pit	StL	SnD	SnF
Atlanta	0	702	454	842	2396	1196	864	772	714	554	2363	2679
Chicago	702	0	324	1093	2136	764	845	764	459	294	2184	2187
Cinci.	454	324	0	1137	2180	798	664	572	284	338	2228	2463
Houston	842	1093	1137	0	1616	1857	1706	1614	1421	799	1521	2021
L.A.	2396	2136	2180	1616	0	2900	2844	2752	2464	1842	95	405
Montreal	1196	764	798	1857	2900	0	396	424	514	1058	2948	2951
New York	864	845	664	1706	2844	396	0	92	386	1002	2892	3032
Phildpha.	772	764	572	1614	2752	424	92	0	305	910	2800	2951
Pittsbrg.	714	459	284	1421	2464	514	386	305	0	622	2512	2646
St. Louis	554	294	338	799	1842	1058	1002	910	622	0	1890	2125
San Diego	2363	2184	2228	1521	95	2948	2892	2800	2512	1890	0	500
San Fran.	2679	2187	2463	2021	405	2951	3032	2951	2646	2125	500	0

Solution

We will illustrate the subtour elimination approach, exploiting the fact that the distance matrix is symmetric. Define the decision variables:

> Y_{ij} = 1 if the link between cities i and j is used, regardless of the direction of travel; 0 otherwise.

Thus, Y(CHI, ATL) = 1 if the link between Chicago and Atlanta is used. Each city or node must be connected to two links. In words, the formulation is:

> Minimize the cost of links selected
> subject to:
> > For each city, the number of links connected to it that are selected = 2
> > Each link can be selected at most once.

The LINGO formulation is shown below:

```
MODEL:
 SETS:
  CITY;
  ROUTE(CITY, CITY)|&1 #GT# &2:COST, Y;
 ENDSETS
 DATA:
  CITY=
     ATL  CHI  CIN HOU   LA  MON   NY  PHI  PIT  STL  SD  SF;
   COST=
     702
     454  324
     842 1093 1137
    2396 2136 2180 1617
    1196  764  798 1857 2900
     864  845  664 1706 2844  396
     772  764  572 1614 2752  424   92
     714  459  284 1421 2464  514  386  305
     554  294  338  799 1842 1058 1002  910  622
    2363 2184 2228 1521   95 2948 2892 2800 2512 1890
    2679 2187 2463 2021  405 2951 3032 2951 2646 2125 500;
 ENDDATA
  MIN = @SUM( ROUTE: Y * COST);
  @SUM( CITY( I)|I #GE# 2: Y(I, 1)) = 2;
  @FOR( CITY( J)|J #GE# 2: @SUM(CITY(I)| I #GT# J:
    Y(I, J)) + @SUM(CITY(K)|K #LT# J: Y(J, K))=2);
  @FOR( ROUTE: Y <= 1);
END
```

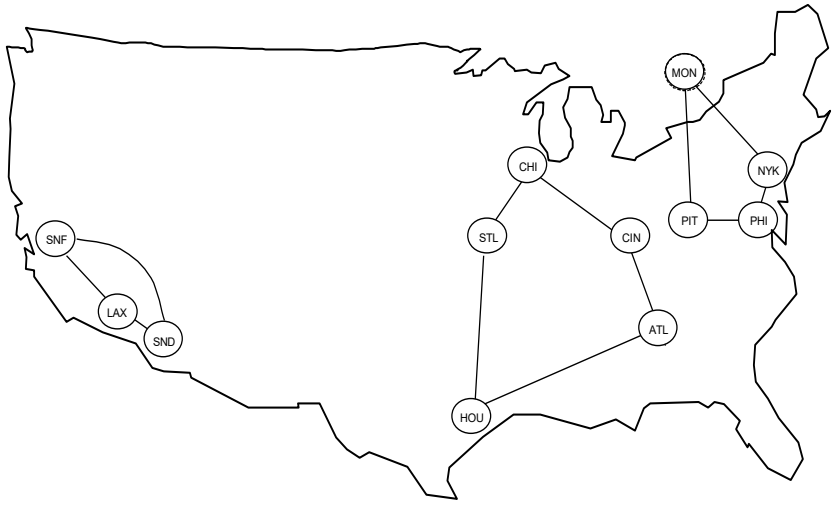
When this model is solved *as an LP*, we get the solution:

```
Optimal solution found at step:       105
Objective value:                 5020.000

    Variable           Value
Y( CIN, ATL)        1.000000
Y( CIN, CHI)        1.000000
Y( HOU, ATL)        1.000000
Y( NYK, MON)        1.000000
Y( PHI, NYK)        1.000000
Y( PIT, MON)        1.000000
Y( PIT, PHI)        1.000000
Y( STL, CHI)        1.000000
Y( STL, HOU)        1.000000
Y( SND, LAX)        1.000000
Y( SNF, LAX)        1.000000
Y( SNF, SND)        1.000000
```

This has a cost of 5020 miles. Graphically, it corresponds to Figure 11.7.

Figure 11.7



Unfortunately, the solution has three subtours. We would like to cut off the smallest subtour by adding the constraint that looks like:

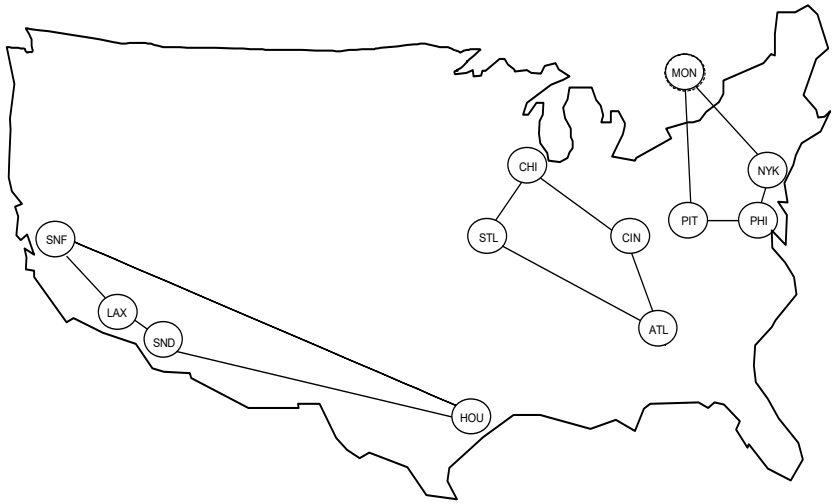
```
!SUBTOUR ELIMINATION;
      Y( SNF, LAX) + Y( SND, LAX) + Y( SNF, SND) <= 2;
```

LINGO, however, works only with numeric subscripts, so if we want to use subscripts like SNF and LAX, we have to first tell LINGO their index values. The follow statements in the LINGO model equations will do this.

```
! A Trick: To make it easier to add problem specific cuts, give ourselves
  some constants equal to index number of city with same name;
    ATL=1; CHI=2; CIN=3; HOU=4;  LAX=5;  MON=6;
    NYK=7; PHI=8; PIT=9; STL=10; SND=11; SNF=12;
! A longer, less clever approach uses the @INDEX() function.
  E.g., LAX = @INDEX(LAX) would achieve the same effect.
   Now we can add cuts, using names directly.;
```

Now, when we solve it *as an LP,* we get a solution with cost 6975, corresponding to Figure 11.8:

Figure 11.8



We cut off the subtour in the southwest by appending the constraint that says at most 3 arcs can be used involving the cities HOU, LAX, SND, and SNF:

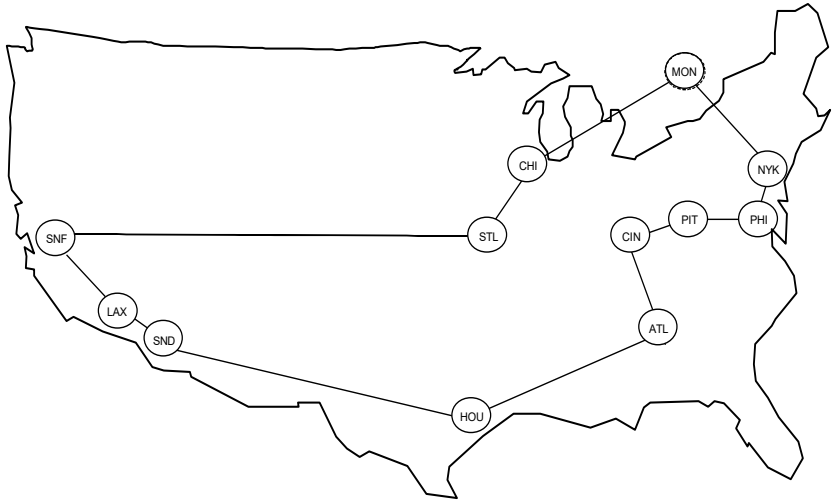
```
  Y(LAX, HOU) + Y(SND, HOU) + Y(SNF, HOU)
+ Y(SND, LAX) + Y(SNF, LAX) + Y(SNF, SND)<= 3;
```

We continue in this fashion appending subtour elimination cuts:

```
Y(NYK, MON) + Y(PHI, MON) + Y(PIT, MON) +
Y(PHI, NYK) + Y(PIT, NYK) + Y(PIT, PHI) <= 3;
Y(NYK, MON) + Y(PHI, MON) + Y(PHI, NYK) <= 2;
```

After the above are all appended, we get the solution shown in Figure 11.9. It is a complete tour with cost $7,577.

Figure 11.9



Note only LPs were solved. No branch-and-bound was required, although in general branching may be required.

Could P. Rose have done as well by trial and error? The most obvious heuristic is the "closest unvisited city" heuristic. If one starts in Cincinnati and next goes to the closest unvisited city at each step and finally returns to Cincinnati, the total distance is 8015 miles, about 6% worse than the optimum.

The Optional Stop TSP

If we drop the requirement that every stop must be visited, we then get the optional stop TSP. This might correspond to a job sequencing problem where v_j is the profit from job j if we do it and c_{ij} is the cost of switching from job i to job j. Let:

$y_j = 1$ if city j is visited, 0 otherwise.

If v_j is the value of visiting city j, then the objective is:

$$\text{Minimize} \sum_i \sum_j c_{ij} x_{ij} - \sum v_j y_j .$$

The constraint sets are:

(1) Each city j can be visited at most once

$$\sum_{i \neq j} x_{ij} = y_j$$

(2) If we enter city j, then we must exit it:

$$\sum_{k \neq j} x_{jk} = y_j$$

(3) No subtours allowed for each subset, S, of cities not including the home base 1.

$\sum_{i,j \in S} x_{ij} \leq |S| - 1$, where $|S|$ is the size of S.

For example, if there are n cities, including the home base, then there are $(n-1)(n-2)/(3 \times 2)$ subsets of size 3.

(4) Alternatively, (3) may be replaced by

$u_j \geq u_i + 1 - (1 - x_{ij})n \quad$ for $j = 2, 3, \ldots, n$.

Effectively, u_j is the sequence number of city j in its tour. Constraint set (3) is much tighter than (4).

11.6.3 Capacitated Multiple TSP/Vehicle Routing Problems

An important practical problem is the routing of vehicles from a central depot, the so-called Vehicle Routing Problem (VRP). An example is the routing of delivery trucks for a parcel delivery service. You can think of this as a multiple traveling salesperson problem with finite capacity for each salesperson. This problem is sometimes called the LTL(Less than TruckLoad) routing problem because a typical recipient receives less than a truck load of goods. A formulation is:

Given:

V = capacity of a vehicle
d_j = demand of city or stop j

Each city, j, must be visited once for $j > 1$:

$$\sum_i x_{ij} = 1$$

Each city $i > 1$, must be exited once:

$$\sum_i x_{ij} = 1$$

No subtours:

$$\sum_{i,j \in s} x_{ij} \leq |S| - 1,$$

No overloads: For each set of cities T, including 1, which constitute more than a truckload:

$$\sum_{i,j \in T} x_{ij} \leq |T| - k,$$

where k = minimum number of cities that must be dropped from T to reduce it to one load.

This formulation can solve to optimality modest-sized problems of say, 25 cities. For larger or more complicated practical problems, the heuristic method of Clarke and Wright (1964) is a standard starting point for quickly finding good, but not necessarily optimal, solutions.

The following is a generic LINGO model for vehicle routing problems:

```
MODEL:    ! (VROUTE);
```

```
! The Vehicle Routing Problem (VRP) occurs in many service
systems such as delivery, customer pick-up, repair and
maintenance. A fleet of vehicles, each with fixed
capacity, starts at a common depot and returns to the
depot after visiting locations where service is demanded.
Problems with more than a dozen cities can take lots of
time.

This instance involves delivering the required amount of
goods to 9 cities from a depot at city 1;

SETS:
  CITY/ Chi Den Frsn Hous KC LA Oakl Anah Peor Phnx/: Q, U;
  ! Q(I) = amount required at city I(given),
     must be delivered by just 1 vehicle.
U(I) = accumulated deliveries at city I ;

  CXC( CITY, CITY): DIST, X;
  ! DIST(I,J) = distance from city I to city J
     X(I,J) is 0-1 variable,
     = 1 if some vehicle travels from city I to J,
     else 0 ;
ENDSETS
DATA:
  ! city 1 represents the common depot, i.e. Q( 1) = 0;
Q= 0    6    3    7    7   18    4    5    2    6;

  ! distance from city I to city J is same from J to I,
  distance from city I to the depot is 0,
  because vehicle need not return to the depot ;
  DIST=  ! To City;
  !Chi  Den Frsn Hous   KC  LA Oakl Anah  Peor Phnx  From;
     0  996 2162 1067  499 2054 2134 2050  151 1713!  Chicago;
     0    0 1167 1019  596 1059 1227 1055  904  792!  Denver;
     0 1167    0 1747 1723  214  168  250 2070  598!  Fresno;
     0 1019 1747    0  710 1538 1904 1528  948 1149!  Houston;
     0  596 1723  710    0 1589 1827 1579  354 1214!  K. City;
     0 1059  214 1538 1589    0  371   36 1943  389!  L. A.;
     0 1227  168 1904 1827  371    0  407 2043  755!  Oakland;
     0 1055  250 1528 1579   36  407    0 1933  379!  Anaheim;
     0  904 2070  948  354 1943 2043 1933    0 1568!  Peoria;
     0  792  598 1149 1214  389  755  379 1568    0;! Phoenix;
  ! VCAP is the capacity of a vehicle ;
  VCAP = 18;
ENDDATA
  !-----------------------------------------------------------;
  ! The objective is to minimize total travel distance;
  MIN = @SUM( CXC: DIST * X);

  ! for each city, except depot....;
  @FOR( CITY( K)| K #GT# 1:

    ! a vehicle does not travel inside itself,...;
  X( K, K) = 0;

  ! a vehicle must enter it,... ;
  @SUM( CITY( I)| I #NE# K #AND# ( I #EQ# 1 #OR#
    Q( I) + Q( K) #LE# VCAP): X( I, K)) = 1;
```

```
  ! a vehicle must leave it after service ;
 @SUM( CITY( J)| J #NE# K #AND# ( J #EQ# 1 #OR#
 Q( J) + Q( K) #LE# VCAP): X( K, J)) = 1;
 ! U( K) = amount delivered on trip up to city K
 >= amount needed at K but <= vehicle capacity;
 @BND( Q( K), U( K), VCAP);
 ! If K follows I, then can bound U( K) - U( I);
 @FOR( CITY( I)| I #NE# K #AND# I #NE# 1: U( K) >=
   U( I) + Q( K) - VCAP + VCAP*( X( K, I) + X( I, K))
   - ( Q( K) + Q( I)) * X( K, I);
 );
 ! If K is 1st stop, then U( K) = Q( K);
 U( K) <= VCAP - ( VCAP - Q( K)) * X( 1, K);
 ! If K is not 1st stop...;
 U( K) >=
 Q( K)+ @SUM( CITY( I)| I #GT# 1: Q( I) * X( I, K));
 );
 ! Make the X's binary;
 @FOR( CXC( I, J): @BIN( X( I, J)) ;);
 ! Must send enough vehicles out of depot;
 @SUM( CITY( J)| J #GT# 1: X( 1, J)) >=
 @FLOOR((@SUM( CITY( I)| I #GT# 1: Q( I))/ VCAP) + .999);
END
```

```
 Optimal solution found at step:       973
 Objective value:                  6732.000
                     Variable           Value
                X( CHI, HOUS)        1.000000
                  X( CHI, LA)        1.000000
                X( CHI, PEOR)        1.000000
                X( CHI, PHNX)        1.000000
                 X( DEN, CHI)        1.000000
               X( FRSN, OAKL)        1.000000
                X( HOUS, CHI)        1.000000
                  X( KC, DEN)        1.000000
                  X( LA, CHI)        1.000000
                X( OAKL, CHI)        1.000000
               X( ANAH, FRSN)        1.000000
                 X( PEOR, KC)        1.000000
               X( PHNX, ANAH)        1.000000
```

By following the links, you can observe that the trips are:

Chicago - Houston;
Chicago - LA;
Chicago - Peoria - KC - Denver;
Chicago - Phoenix - Anaheim - Fresno - Oakland.

The solvability of practical VRP's depends upon a variety of typical complications: a) average number of stops per vehicle: 2 or 3 stops/vehicle is easily solved. Unlimited stops is essentially the Traveling Sales Person problem, which is moderately easy to solve; b) number of vehicle types and limits on the number of each. All identical vehicles is the easier; c) number of dimensions to capacity. Just 1, e.g., just a weight limit, is easier, but in practice there may also be constraints on volume (cube), total drive time, etc.; d) time windows. If there are limits on when each customer can be visited, then the problem may be a lot more difficult; e) sparsity of the distance matrix. If many of the possible arcs are prohibited, this tends to make the problem easier; f) static distance matrix. If the travel time on an arc depends upon the time of day, this makes the problem more difficult; g) split deliveries. If the demand at a customer is greater than vehicle capacity, then a split delivery is unavoidable. If split deliveries are optional, then this may reduce the total distance in some instances; h) symmetric distance matrix. If it is symmetric, this may make the problem slightly easier; i) geometry of the region. VRP's in Chile are much easier to solve than VRP's in the U.S.; j) number of depots. It is typically 1, but if there is additionally the choice of which depot serves which customer, that may make the problem harder.

Combined DC Location/Vehicle Routing

Frequently, there is a vehicle routing problem associated with opening a new plant or distribution center (DC). Specifically, given the customers to be served from the DC, what trips are made, so as to serve the customers at minimum cost. A "complete" solution to the problem would solve the location and routing problems simultaneously. The following IP formulation illustrates one approach:

Parameters

F_i = fixed cost of having a DC at location i,
C_j = cost of using route j,
a_{ijk} = 1 if route j originates at DC i and serves customer k. There is exactly one DC associated with each route.

Decision variables

$y_i = 1$ if we use DC i, else 0,
$x_j = 1$ if we use route j, else 0

The Model

Minimize $\sum_i F_i y_i + \sum_j c_j x_j$

subject to

(Demand constraints)
For each customer k:

$$\sum_i \sum_j a_{ijk} x_j = 1$$

(Forcing constraints)
For each DC i and customer k:

$$\sum_j a_{ijk} x_j \le y_i$$

11.6.4 Minimum Spanning Tree

A spanning tree of n nodes is a collection of $n - 1$ arcs, so there is exactly one path between every pair of nodes. A minimum cost spanning tree might be of interest, for example, in designing a communications network.

Assume node 1 is the root of the tree. Let $x_{ij} = 1$ if the path from 1 to j goes through node i immediately before node j, else $x_{ij} = 0$.

A formulation is:

Minimize $\sum_i \sum_j c_{ij}x_{ij}$

subject to

(1) $\sum_i \sum_j x_{ij} = n - 1$,

(2) $\sum_{i,j \in S} x_{ij} \leq |S| - 1$ for every strict subset S of $\{1, 2,\ldots,n\}$,

$x_{ij} = 0$ or 1.

An alternative to (1) and (2) is the following set of constraints based on assigning a unique sequence number u_j to each node:

$$\sum_{i \neq j} x_{ij} = 1, \text{ for } j = 2, 3, 4,\ldots,n,$$

$u_j \geq u_i + x_{ij} - (n-2)(1 - x_{ij}) + (n-3)x_{ji}$, for $j = 2, 3, 4, \ldots, n$.

$u_j \geq 0$.

In this case, u_j is the number of arcs between node j and node 1. A numeric example of the sequence numbering formulation is in section 8.9.8.

If one has a pure spanning tree problem, then the "greedy" algorithm of Kruskal (1956) is a fast way of finding optimal solutions.

11.6.5 The Linear Ordering Problem

A problem superficially similar to the TSP is the linear ordering problem. One wants to find a strict ordering of n objects. Applications are to ranking in sports tournaments, product preference ordering in marketing, job sequencing on one machine, ordering of industries in an input-output matrix, ordering of historical objects in archeology, and others. See Grötschel et al. (1985) for a further discussion. The linear ordering problem is similar to the approach of conjoint analysis sometimes used in marketing. The crucial input data are cost entries c_{ij}. If object i appears *anywhere* before object j in the proposed ordering, then c_{ij} is the resulting cost. The decision variables are:

$x_{ij} = 1$ if object i precedes object j, either directly or indirectly for all $i \neq j$.

The problem is:

Minimize $\sum_i \sum_j c_{ij} x_{ij}$

subject to

(1) $x_{ij} + x_{ji} = 1$ for all $i \neq j$

If i precedes j and j precedes k, then we want to imply that i precedes k. This is enforced with the constraints:

(2) $x_{ij} + x_{jk} + x_{ki} \leq 2$ for all i, j, k with $i \neq j$, $i \neq k$, $j \neq k$.

The size of the formulation can be cut in two by noting that $x_{ji} = 1 - x_{ij}$. Thus, we substitute out x_{ji} for $j > i$. Constraint set (1) becomes simply $0 \le x_{ij} \le 1$. Constraint set (2) becomes:

(2') $x_{ij} + x_{jk} - x_{ik} + s_{ijk} = 1$ for all $i < j < k$
$0 \le s_{ijk} \le 1$

There are $n!/((n-3)!\,3!) = n \times (n-1) \times (n-2)/6$ ways of choosing 3 objects from n, so the number of constraints is approximately $n^3/6$.

Example
Ten Czech, German, and North American beverages were subject to taste tests by unbiased German testers. Each of the $10 \times 9/2 = 45$ possible pairs was subject to a taste test by six judges. The element $C(I, J)$ in the C matrix in the model below is the number of times out of six beverage I was preferred to beverage J. If we want to have a complete ranking for the beverages, a reasonable objective is to maximize the number of pairwise comparisons for which our ranking agrees with the pairwise ranking of the judges:

```
MODEL:
! Linear ordering of objects or products,
   based on pairwise comparisons(LINERORD);
SETS:
 PROD: RANK; ! Each product will get a rank;
 PXP( PROD, PROD): C;
ENDSETS
DATA:
 PROD = KONIG, FURST, PILSURQ, GUNZB, RIEGELE,
        PAULA, JEVER, BECKS,   WARST, BUD;
! Some data on German beverages;
 C= ! Times that object I was preferred over J;
0    2    2    3    3    5    5    5    4    4
4    0    3    3    4    3    2    3    2    2
4    3    0    3    5    4    3    2    4    4
3    3    3    0    5    6    3    4    4    3
3    2    1    1    0    1    4    4    5    3
1    3    2    0    5    0    5    4    1    4
1    4    3    3    2    1    0    2    1    3
1    3    4    2    2    2    4    0    4    2
2    4    2    2    1    5    5    2    0    4
2    4    2    3    3    2    3    4    2    0;
ENDDATA
!-----------------------------------------------;
SETS:
PIP( PROD, PROD)| &1 #LT# &2:
   X; ! X(I,J) = 1 if I precedes J in our ranking;
PIPIP( PROD, PROD, PROD)
             | &1 #LT# &2 #AND# &2 #LT# &3: S;
ENDSETS
! Maximize the number of times our pairwise
  ordering matches that of our testers;
MAX =
@SUM( PIP( I, J): C( I, J) * X( I, J)
      + C( J, I) *(1 - X( I, J)));
```

```
! The rankings must be transitive, that is,
  If I->J and J->K, then I->K;
@FOR( PIPIP( I, J, K):
!   Note N*(N-1)*(N-2)/6 of these!;
X( I, J) + X ( J, K) - X( I, K)
                     + S( I, J, K) = 1;
    @BND( 0, S( I, J, K), 1);
   );
@FOR( PIP: @BIN( X);); ! Make X's 0 or 1;

! Count number products before product I( + 1);
@FOR( PROD( I):
 RANK( I) = 1 + @SUM( PIP( K, I): X( K, I))
              + @SUM( PIP( I, K): 1 - X( I, K));
   );
END
```

When solved, we get an optimal objective value of 168. This means out of the (10 * 9/2)* 6 = 270 pairwise comparisons, the pairwise rankings agreed with LINGO's complete ranking 168 times:

```
Optimal solution found at step:         50
Objective value:                   168.0000
Branch count:                             0

      Variable           Value        Reduced Cost
  RANK( KONIG)        3.000000           0.0000000
  RANK( FURST)        10.00000           0.0000000
RANK( PILSURQ)        2.000000           0.0000000
  RANK( GUNZB)        1.000000           0.0000000
RANK( RIEGELE)        7.000000           0.0000000
  RANK( PAULA)        5.000000           0.0000000
  RANK( JEVER)        9.000000           0.0000000
  RANK( BECKS)        8.000000           0.0000000
  RANK( WARST)        4.000000           0.0000000
    RANK( BUD)        6.000000           0.0000000
```

According to this ranking, *GUNZB* comes out number 1 (most preferred), while *FURST* comes out tenth (least preferred). It is important to note that there may be alternate optima. This means there may be alternate orderings, all of which match the input pairings 168 times out of 270. In fact, you can show that there is another ordering with a value of 168 in which *PILSURQ* is ranked first.

11.6.6 Quadratic Assignment Problem

The quadratic assignment problem has the same constraint set as the linear assignment problem. However, the objective function contains products of two variables. Notationally, it is:

$$\text{Min} \quad \sum_i \sum_j \sum_k \sum_l c_{ijkl} x_{ij} x_{kl}$$

subject to:

For each j:

$$\sum_i x_{ij} = 1$$

For each i:

$$\sum_j x_{ij} = 1$$

Some examples of this problem are:

(a) *Facility layout*. If d_{jl} is the physical distance between room j and room l; s_{ik} is the communication traffic between department i and k; and $x_{ij} = 1$ if department i is assigned to room j, then we want to minimize:

$$\sum_i \sum_j \sum_k \sum_l x_{ij} x_{kl} d_{jl} s_{ik}$$

(b) *Vehicle to gate assignment at a terminal*. If d_{jl} is the distance between gate j and gate l at an airline terminal, passenger train station, or at a truck terminal; s_{ik} is the number of passengers or tons of cargo that needs to be transferred between vehicle i and vehicle k; and $x_{ij} = 1$ if vehicle i (incoming or outgoing) is assigned to gate j, then we again want to minimize:

$$\sum_i \sum_j \sum_k \sum_l x_{ij} x_{kl} d_{jl} s_{ik}$$

(c) *Radio frequency assignment*. If d_{ij} is the physical distance between transmitters i and j; s_{kl} is the distance in frequency between k and l; and p_i is the power of transmitter i, then we want $c_{ijkl} = \max\{p_i, p_j\}\ (1/d_{ij})(1/s_{kl})$ to be small if transmitter i is assigned frequency k and transmitter j is assigned frequency l.

(d) *VLSI chip layout*. The initial step in the design of a VLSI (very large scale integrated) chip is typically to assign various required components to various areas on the chip. See Sarrafzadeh and Wong (1996) for additional details. Steinberg (1961) describes the case of assigning electronic components to a circuit board, so as to minimize the total interconnection wire length. For the chip design case, typically the chip area is partitioned into 2 to 6 areas. If d_{jl} is the physical distance between area j and area l; s_{ik} is the number of connections required between components i and k; and $x_{ij} = 1$ if component i is assigned to area j, then we again want to minimize:

$$\sum_i \sum_j \sum_k \sum_l x_{ij} x_{kl} d_{jl} s_{ik}$$

(e) *Disk file allocation*. If w_{ij} is the interference if files i and j are assigned to the same disk, we want to assign files to disks, so total interference is minimized.

(f) *Type wheel design*. Arrange letters and numbers on a type wheel, so (a) most frequently used ones appear together and (b) characters that tend to get typed together (e.g., q u) appear close together on the wheel.

The quadratic assignment problem is a notoriously difficult problem. If someone asks you to solve such a problem, you should make every effort to show the problem is not really a quadratic assignment problem. One indication of its difficulty is the solution is not naturally integer.

One of the first descriptions of quadratic assignment problems was by Koopmans and Beckmann (1957). For this reason, this problem is sometimes known as the Koopmans-Beckmann problem. They illustrated the use of this model to locate interacting facilities in a large country. Elshafei (1977) illustrates the use of this model to lay out a hospital. Specifically, 19 departments are assigned to 19 different physical regions in the hospital. The objective of Elshafei was to minimize the total distance patients had to walk between departments. The original assignment used in the hospital required a distance of 13,973,298 meters per year. An optimal assignment required a total distance of 8,606,274 meters. This is a reduction in patient travel of over 38%.

Small quadratic assignment problems can be converted to linear integer programs by the transformation:

Replace the product $x_{ij}\, x_{kl}$ by the single variable z_{ijkl}. The objective is then:

$$\text{Min} \quad \sum_i \sum_j \sum_k \sum_l c_{i\,j\,k\,l}\, z_{i\,j\,k\,l}$$

Notice if there are N departments and N locations, then there are $N \times N$ variables of type x_{ij}, and $N \times N \times N \times N$ variables of type z_{ijkl} variables. This formulation can get large quickly. Several reductions are possible:

1) The terms $c_{ijkl}\, x_{ij}\, x_{kl}$ and $c_{klij}\, x_{kl}\, x_{ij}$ can be combined into the term:

   $$(c_{ijkl} + c_{klij})\, x_{kl}\, x_{ij}$$

   to reduce the number of z variables and associated constraints needed by a factor of 2.

2) Certain assignments can be eliminated beforehand (e.g., a large facility to a small location). Many of the cross terms, c_{ijkl} , are zero (e.g., if there is no traffic between facility i and facility k), so the associated z variables need not be introduced.

The non-obvious thing to do now is to ensure that $z_{ijkl} = 1$ if and only if both x_{ij} and $x_{kl} = 1$. Sherali and Adams(1999) point out that constraints of the following type will enforce this requirement:

For a given i, k, l:

$$x_{kl} = \sum_{j, j \neq l} z_{ijkl}$$

In words, if object k is assigned to location l, then for any other object i, $i \neq k$, there must be some other location j, $j \neq l$, to which i is assigned.

The following is a LINGO implementation of the above for deciding which planes should be assigned to which gates at an airport, so that the distance weighted cost of changing planes for the passengers is minimized:

```
MODEL:
! Quadratic assignment problem(QAP006);
!  Given number of transfers between flights,
   distance between gates,
   assign flights to gates to minimize total transfer cost;
 SETS:
  FLIGHT/1..6/;
  GATE/ E3 E4 E5 F3 F4 F5/;! Gates at terminal 2 of O'Hare;
  GXG( GATE, GATE)| &1 #LT# &2: T; ! Inter gate times(symmetric);
  FXF( FLIGHT, FLIGHT)| &1 #LT# &2: N; ! Transfers between flights;
  FXG( FLIGHT, GATE):   X; ! Flight to gate assignment variable;
 ENDSETS
 DATA:
  T =  70  40  60  90  90  ! Time between gates;
          50 100  80 110
             100  90 130
                  60  40
                      30;
  N =  12   0  12   0   5
          30  35  20  13  ! No. units between flights;
              40  20  10
                   0   6
                      14;
 ENDDATA
!-----------------------------------------------------------;
! Warning: may be very slow for no. objects > 7;
  SETS:  ! Warning: this set gets big fast!;
   TGTG( FLIGHT, GATE, FLIGHT, GATE)| &1 #LT# &3: Z;
  ENDSETS
! Min the cost of transfers * distance;
   MIN = @SUM( TGTG( B, J, C, K)| J #LT# K:
            Z( B, J, C, K) * N( B, C) * T( J, K))
       + @SUM( TGTG( B, J, C, K)| J #GT# K:
            Z( B, J, C, K) * N( B, C) * T( K, J));
! Each flight, B, must be assigned to a gate;
    @FOR( FLIGHT( B):
      @SUM( GATE( J): X( B, J)) = 1;
        );
! Each gate, J, must receive one flight;
    @FOR( GATE( J):
      @SUM( FLIGHT( B): X( B, J)) = 1;
        );
! Make the X's binary;
    @FOR( FXG: @BIN( X);
        );
```

```
! Force the Z() to take the correct value relative to the X();
  @FOR( FXG( C, K):
    @FOR( GATE( J)| J #NE# K:
! If C is assigned to K,  some B must be assigned to J...;
      X( C, K) = @SUM( TGTG( B, J, C, K)| B #NE# C : Z( B, J, C, K))
               + @SUM( TGTG( C, K, B, J)| B #NE# C : Z( C, K, B, J));
        );
    @FOR( FLIGHT( B)| B #NE# C:
!  and B must be assigned to some J;
      X( C, K) = @SUM( TGTG( B, J, C, K)| J #NE# K : Z( B, J, C, K))
               + @SUM( TGTG( C, K, B, J)| J #NE# K : Z( C, K, B, J));
        );
     );
 END
```

The solution is:

```
Global optimal solution found at step:          1258
Objective value:                            13490.00
Branch count:                                      0

                Variable           Value
              X( 1, E4)        1.000000
              X( 2, F4)        1.000000
              X( 3, F3)        1.000000
              X( 4, F5)        1.000000
              X( 5, E3)        1.000000
              X( 6, E5)        1.000000
```

Thus, flight 1 should be assigned to gate *E4*, flight 2 to gate *F4*, etc. The total passenger travel time in making the connections will be 13,490. Notice that this formulation was fairly tight. No branches were required to get an integer solution from the LP solution.

11.7 Problems of Grouping, Matching, Covering, Partitioning, and Packing

There is a class of problems that have the following essential structure:

1) There is a set of *m* objects, and
2) They are to be grouped into subsets, so some criterion is optimized.

Some example situations are:

	Objects	Group	Criteria for a Group
(a)	Dormitory inhabitants	Roommates	At most two to a room; no smokers with nonsmokers.
(*b*)	Deliveries to customers	Trip	Total weight assigned to trip is less-than-or-equal-to vehicle capacity. Customers in same trip are close together.
(*c*)	Sessions at a scientific meeting	Sessions scheduled for same time slot	No two sessions on same general topic. Enough rooms of sufficient size.
(*d*)	Exams to be scheduled	Exams scheduled for same time slot	No student has more than one exam in a given time slot.
(*e*)	Sportsmen	Foursome (e.g., in golf or tennis doubles).	Members are of comparable ability, appropriate combination of sexes as in tennis mixed doubles.
(*f*)	States on map to be colored.	All states of a given color	States in same group/color cannot be adjacent.
(*g*)	Finished good widths needed in a paper plant	Widths cut from a single raw paper roll.	Sum of finished good widths must not exceed raw material width.
(h)	Pairs of points to connect on a circuit board	Connection layers underneath the circuit board	Connection paths in a layer should not intersect. Total lengths of paths are small.
(i)	Financial instruments, e.g., mortgages	Package of instruments, e.g., mortgage backed securities	Package must be approximately of a target size, target credit worthiness, target interest rate.

If each object can belong to at most one group, it is called a *packing* problem. For example, in a delivery problem, as in (*ii*) above, it may be acceptable that a low priority customer not be included in any trip today if we are confident the customer could be almost as well served by a delivery tomorrow. If each object must belong to exactly one group, it is called a *partitioning* problem. For example, in circuit board routing as in (*vii*) above, if a certain pair of points must be connected, then that pair of points must be assigned to exactly one connection layer underneath the board. If each object must belong to at least one group, it is called a *covering* problem. A packing or partitioning problem with group sizes limited to two or less is called a *matching* problem. Specialized and fast algorithms exist for matching problems. A problem closely related to covering problems is the cutting stock problem. It arises in paper, printing, textile, and steel industries. In this problem, we want to determine cutting patterns to be used in cutting up large pieces of raw material into finished-good-size pieces.

Although grouping problems may be very easy to state, it may be very difficult to find a provably optimal solution if we take an inappropriate approach. There are two common approaches to formulating

grouping problems: (1) assignment style, or (2) the partition method. The former is convenient for small problems, but it quickly becomes useless as the number of objects gets large.

11.7.1 Formulation as an Assignment Problem

The most obvious formulation for the general grouping problem is based around the following definition 0/1 decision variables:

$X_{ij} = 1$ if object j is assigned to group i, 0 otherwise.

A drawback of this formulation is that it has a lot of symmetry. There are many alternate optimal solutions. All of which essentially are identical. For example, assigning golfers *A*, *B*, *C*, and *D* to group 1 and golfers *E*, *F*, *G*, and *H* to group 2 is essentially the same as assigning golfers *E*, *F*, *G*, and *H* to group 1 and golfers *A*, *B*, *C* and *D* to group 2. These alternate optima make the typical integer programming algorithm take much longer than necessary.

We can eliminate this symmetry and the associated alternate optima with no loss of optimality if we agree to the following restrictions: (a) object 1 can only be assigned to group 1; (b) object 2 can only be assigned to groups 1 or 2 and only to 1 if object 1 is also assigned to 1; (c) and in general, object j can be assigned to group $i < j$, only if object i is also assigned to group i. This implies in particular that:

$X_{ii} = 1$, if and only if object i is the lowest indexed object in its group, and
X_{ij} is defined only for $i \le j$.

Now we will look at several examples of grouping problems and show how to solve them.

11.7.2 Matching Problems, Groups of Size Two

The roommate assignment problem is a simple example of a grouping problem where the group size is two. An example of this is a problem solved at many universities at the start of the school year before the first-year or freshman students arrive. The rooms in a freshman dormitory typically take exactly two students. How should new incoming students be paired up? One approach that has been used is that for every possible pair of students, a score is calculated which is a measure of how well the school thinks this particular pair of students would fare as roommates. Considerations that enter into a score are things such as: a smoker should not be matched with a nonsmoker, a person who likes to study late at night should not be paired with a student who likes to get up early and study in the morning. Let us suppose we have computed the scores for all possible pairs of the six students: Joe, Bob, Chuck, Ed, Evan, and Sean. A scaler model for this problem might be:

```
! Maximize total score of pairs selected;
 MAX= 9*X_JOE_BOB   + 7*X_JOE_CHUCK   + 4*X_JOE_ED
    + 6*X_JOE_EVAN + 3*X_JOE_SEAN    + 2*X_BOB_CHUCK
    + 8*X_BOB_ED      + X_BOB_EVAN    + 7*X_BOB_SEAN
    + 3*X_CHUCK_ED + 4*X_CHUCK_EVAN + 9*X_CHUCK_SEAN
    + 5*X_ED_EVAN   + 5*X_ED_SEAN     + 6*X_EVAN_SEAN;

 ! Each student must be in exactly one pair;
 [JOE]   X_JOE_BOB   + X_JOE_CHUCK + X_JOE_ED
      + X_JOE_EVAN + X_JOE_SEAN = 1;
 [BOB]   X_JOE_BOB + X_BOB_CHUCK + X_BOB_ED
      + X_BOB_EVAN+ X_BOB_SEAN = 1;
 [CHUCK]   X_JOE_CHUCK + X_BOB_CHUCK + X_CHUCK_ED
```

```
        + X_CHUCK_EVAN+ X_CHUCK_SEAN = 1;
 [ED]  X_JOE_ED + X_BOB_ED + X_CHUCK_ED + X_ED_EVAN
      + X_ED_SEAN =    1;
 [EVAN]  X_JOE_EVAN + X_BOB_EVAN + X_CHUCK_EVAN + X_ED_EVAN
      + X_EVAN_SEAN =    1;
 [SEAN]  X_JOE_SEAN + X_BOB_SEAN + X_CHUCK_SEAN
       + X_ED_SEAN + X_EVAN_SEAN =    1;

 ! Assignments must be binary, not fractional;
  @BIN( X_JOE_BOB);   @BIN( X_JOE_CHUCK);   @BIN( X_JOE_ED);
  @BIN( X_JOE_EVAN);  @BIN( X_JOE_SEAN);    @BIN( X_BOB_CHUCK);
  @BIN( X_BOB_ED);   @BIN( X_BOB_EVAN);    @BIN( X_BOB_SEAN);
  @BIN( X_CHUCK_ED); @BIN( X_CHUCK_EVAN); @BIN( X_CHUCK_SEAN);
  @BIN( X_ED_EVAN);  @BIN( X_ED_SEAN);    @BIN( X_EVAN_SEAN);
```

Notice that there is a variable `X_JOE_BOB`, but not a variable `X_BOB_JOE`. This is because we do not care whose name is listed first on the door. We only care about which two are paired together. We say we are interested in unordered pairs.

A typical dormitory may have 60, or 600, rather than 6 students, so a general, set based formulation would be useful. The following formulation shows how to do this in LINGO. One thing we want to do in the model is to tell LINGO that we do not care about the order of persons in a pair. LINGO conveniently allows us to put conditions on which of all possible members (pairs in this case) of a set are to be used in a specific model. The key statement in the model is:

```
  PXP( PERSON, PERSON)| &1 #LT# &2: VALUE, X;
```

The fragment, `PXP( PERSON, PERSON)`, by itself, tells LINGO that the set `PXP` should consist of all possible combinations, 6*6 for this example, of two persons. The conditional phrase, `| &1 #LT# &2` , however, tells LINGO to restrict the combinations to those in which the index number, &1, of the first person in a pair should be strictly less than the index number, &2, of the second person.

```
MODEL: ! (roomates.lng);
 SETS:
  PERSON;
! Joe rooms with Bob means the same as
  Bob rooms with Joe, so we need only the
   upper triangle;
  PXP( PERSON, PERSON)| &1 #LT# &2: VALUE, X;
 ENDSETS
 DATA:
  PERSON = Joe  Bob  Chuck  Ed  Evan Sean;
   Value =       9     7     4     6    3   ! Joe;
                       2     8     1    7   ! Bob;
                             3     4    9   ! Chuck;
                                   5    5   ! Ed;
                                        6 ; ! Evan;
```

```
  ENDDATA

! Maximize the value of the matchings;
  MAX = @SUM( PXP(I,J): Value(i,j)* X(I,J));

! Each person appears in exactly one match;
   @FOR( PERSON( K):
        @SUM( PXP(K,J): X(K,J)) + @SUM( PXP(I,K): X(I,K)) = 1;
        );
! No timesharing;
  @FOR( PXP(I,J): @BIN(X(I,J)));
END
```

The constraint, `@SUM( PXP(K,J): X(K,J)) + @SUM( PXP(I,K): X(I,K))= 1` has two terms, the first where student `K` is the first person in the pair, the second summation is over the variables where student `K` is the second person in the pair. For example, in the scaler formulation, notice that `ED` is the first person in two of the pairs, and the second person of three of the pairs.

The following solution, with value 23, is found.

```
                   Variable           Value
                X( JOE, EVAN)        1.000000
                  X( BOB, ED)        1.000000
              X( CHUCK, SEAN)        1.000000
```

So Joe is to be paired with Evan, Bob with Ed, and Chuck with Sean. This model scales up well in that it can be easily solved for large numbers of objects, e.g., many hundreds.

For a different perspective on matching, see the later section on “stable matching”.

11.7.3 Groups with More Than Two Members

The following example illustrates a problem recently encountered by an electricity generating firm and its coal supplier. You are a coal supplier and you have a nonexclusive contract with a consumer owned and managed electric utility, Power to the People (PTTP). You supply PTTP by barge. Your contract with PTTP stipulates that the coal you deliver must have at least 13000 BTU’s per ton, no more than 0.63% sulfur, no more than 6.5% ash, and no more than 7% moisture. Historically, PTTP would not accept a barge if it did not meet the above requirements.

You currently have the following barge loads available.

Barge	BTU/ton	Sulfur%	Ash%	Moisture%
1	13029	0.57	5.56	6.2
2	14201	0.88	6.76	5.1
3	10630	0.11	4.36	4.6
4	13200	0.71	6.66	7.6
5	13029	0.57	5.56	6.2
6	14201	0.88	6.76	5.1
7	13200	0.71	6.66	7.6
8	10630	0.11	4.36	4.6
9	14201	0.88	6.76	5.1
10	13029	0.57	5.56	6.2
11	13200	0.71	6.66	7.6
12	14201	0.88	6.76	5.1

This does not look good. Only barges 1, 5, and 10 satisfy PTTP's requirement. What can we do? Suppose that after reading the fine print of your PTTP contract carefully, you initiate some discussions with PTTP about how to interpret the above requirements. There might be some benefits if you could get PTTP to reinterpret the wording of the contract so that the above requirements apply to collections of up to three barges. That is, if the average quality taken over a set of N barges, N less than four, meets the above quality requirements, then that set of N barges is acceptable. You may specify how the sets of barges are assembled. Each barge can be in at most one set. All the barges in a set must be in the same shipment.

Looking at the original data, we see, even though there are twelve barges, there are only four distinct barge types represented by the original first four barges. In reality, you would expect this: each barge type corresponding to a specific mine with associated coal type.

Modeling the barge grouping problem as an assignment problem is relatively straightforward. The essential decision variable is defined as $X(I, J)$ = number of barges of type I assigned to group J. Note we have retained the convention of not distinguishing between barges of the same type. Knowing there are twelve barges, we can restrict ourselves to at most six groups without looking further at the data. The reasoning is: Suppose there are seven nonempty groups. Then, at least two of the groups must be singletons. If two singletons are feasible, then so is the group obtained by combining them. Thus, we can write the following LINGO model:

```
MODEL:
SETS:
  MINE: BAVAIL;
  GROUP;
  QUALITY: QTARG;
!  Composition of each type of MINE load;
  MXQ( MINE, QUALITY): QACT;
!assignment of which MINE to which group;
!no distinction between types;
  MXG( MINE, GROUP):X;
ENDSETS
DATA:
  MINE =   1..4;
       ! Barges available of each type(or mine);
  BAVAIL = 3 4 2 3;
  QUALITY =  BTU,  SULF, ASH, MOIST;
       ! Quality targets as upper limits;
  QTARG = - 13000  0.63  6.5   7;
       ! Actual qualities of each mine;
  QACT =   -13029  0.57  5.56 6.2
           -14201  0.88  6.76 5.1
           -10630  0.11  4.36 4.6
           -13200  0.71  6.66 7.6;
! We need at most six groups;
  GROUP = 1..6;
  GRPSIZ = 3;
ENDDATA
! Maximize no. of barges assigned;
MAX = @SUM( MXG: X);
! Upper limit on group size;
 @FOR( GROUP(J): @SUM( MINE( I): X(I, J))
       <= GRPSIZ;);
! Assign no more of a type than are available;
 @FOR( MINE(I): @SUM( GROUP( J): X( I, J))
       <= BAVAIL( I));
! The blending constraints for each group;
 @FOR( GROUP(J):
  @FOR( QUALITY ( H):
   @SUM( MINE( I): X( I, J) * QACT( I, H)) <=
   @SUM( MINE( I): X( I, J) * QTARG( H));
      ));
! barges must be integers;
@FOR( MXG: @GIN( X));
END
```

The following solution shows that you can now sell ten barges, rather than three, to PTTP.

```
Objective value:                10.00000
Variable           Value
X( 1, 1)        1.000000
X( 2, 2)        2.000000
X( 3, 2)        1.000000
X( 1, 4)        2.000000
X( 4, 4)        1.000000
X( 2, 5)        2.000000
X( 3, 5)        1.000000
```

For example, group 1 is simply one barge of type 1. Group 2 consists two barges of type 2 and one barge of type 3. The above formulation may not scale well. The actual application typically had about 60 barges in a candidate set. The above formulation may be slow to solve problems of that size. The next section discusses how the partitioning approach can be efficiently used for such problems.

Solving with a Partitioning Formulation

Modest-sized integer programs can nevertheless be very difficult to solve. There are a number of rules that are useful when facing such problems. Two useful rules for difficult integer programs are:

1) Do Not Distinguish the Indistinguishable;

2) Presolve subproblems.

The barge matching example can be solved almost "by hand" with the matching or grouping (as opposed to the assignment) approach. Applying the rule "Presolve subproblems," we can enumerate all feasible combinations of three or less barges selected from the four types. Applying the "Don't distinguish" rule again, we do not have to consider combinations such as (1,1) and (2,2,2), because such sets are feasible if and only if the singleton sets (e.g., (1) and (2)) are also feasible. Thus, disregarding quality, there are four singleton sets, six doubleton sets, four distinct triplets (e.g., (1,2,3)) and twelve paired triplets (e.g., (1,1,2)) for a total of 26 combinations. It is not hard to show, even manually, that the only feasible combinations are (1), (1,1,4), and (2,2,3). Thus, the matching-like IP we want to solve to maximize the number of barges sold is:

```
Max = S001 + 3 * S114 + 3 * S223;
 S001 + 2 * S114         <= 3 ;
      !(No. of type 1 barges);
                2 * S223  <= 4 ;
      !(No. of type 2 barges);
                S223  <= 2 ;
      !(No. of type 3 barges);
              S114        <= 3 ;
      !(No. of type 4 barges);
```

This is easily solved to give *S001* = 1, *S114* = 1, and *S223* = 2, with an objective value of 10.

For the given data, we can ship at most ten barges. One such way of matching them, so each set satisfies the quality requirements is as follows:

	Average Quality of the Set			
Barges in set	**BTU%**	**Sulfur%**	**Ash%**	**Moisture%**
1	13029	0.57	5.56	6.2
4, 5, 10	13086	0.6167	5.927	6.667
2, 3, 6	13010	0.6233	5.96	4.933
8, 9, 12	13010	0.6233	5.96	4.933

This matches our LINGO derived solution.

11.7.4 Groups with a Variable Number of Members, Assignment Version

In many applications of the grouping idea, the group size may be variable. The following example from the financial industry illustrates. A financial services firm has financial objects (e.g., mortgages) it wants to "package" and sell. One of the features of a package is that it must contain a combination of objects whose values total at least one million dollars. For our purposes, we will assume this is the only qualification in synthesizing a package. We want to maximize the number of packages we form. We first give an assignment formulation. The key declaration in this formulation is:

```
OXO( OBJECT, OBJECT)| &1 #LE# &2: X;
```

This implies there will be a variable of the form `X(I,J)` with always the index `I` ≤ `J`. Our interpretation of this variable will be:

`X(I,J)` = 1 means object `J` is assigned to the same group as object `I`, and further,
`X(I,I)` = 1 means object `I` is the lowest indexed object in that group.

```
MODEL:
! Object bundling model. (OBJBUNDL);
!A broker has a number of loans of size from $55,000 to $946,000.
 The broker would like to group the loans into packages
 so that each package has at least $1M in it,
 and the number of packages is maximized;
! Keywords: bundling, financial, set packing;
 SETS:   OBJECT: VALUE, OVER;
   OXO( OBJECT, OBJECT)| &1 #LE# &2: X;
 ENDSETS
 DATA:
  OBJECT= A   B   C   D   E   F   G  H   I   J   K  L   M   N   P   Q   R;
   VALUE=910 870 810 640 550 250 120 95 55 200 321 492 567 837 193 364 946;
! The value in each bundle must be >= PKSIZE;
    PKSIZE = 1000;
 ENDDATA
!-----------------------------------------------;
! Definition of variables;
!  X( I, I) = 1 if object I is lowest numbered
             object in its package;
!  X( I, J) = 1 if object j is assigned to package I;
! Maximize number of packages assembled;
  MAX = @SUM( OBJECT( I): X( I, I));
```

```
  @FOR( OBJECT( K):
! Each object can be assigned to at most one package;
    @SUM( OXO( I, K): X( I, K)) <= 1;
! A package must be at least PSIZE in size;
    @SUM( OXO( K, J): VALUE( J) * X( K, J))
         - OVER( K) = PKSIZE * X( K, K);
      );
! The X( I, J) must = 0 or 1;
  @FOR( OXO( I, J): @BIN( X( I, J))););
END
```

A solution is:

```
Variable           Value
X( A, A)        1.000000
X( A, H)        1.000000
X( B, B)        1.000000
X( B, F)        1.000000
X( C, C)        1.000000
X( C, J)        1.000000
X( D, D)        1.000000
X( D, Q)        1.000000
X( E, E)        1.000000
X( E, L)        1.000000
X( G, G)        1.000000
X( G, K)        1.000000
X( G, M)        1.000000
X( I, I)        1.000000
X( I, R)        1.000000
X( N, N)        1.000000
X( N, P)        1.000000
```

Thus, eight packages are constructed. Namely: *AH*, *BF*, *CJ*, *DQ, EL, IR, JN, GKM*, and *NP*. It happens that every object appears in some package. There are alternate packings of all the objects into eight groups. Thus, one may wish to consider secondary criteria for choosing one alternate optimum over another (e.g., the largest package should be as close as possible to one million in size). The worst package in the fairness sense in the above solution is BF. It is over the target of 1,000,000 by 120,000.

11.7.5 Groups with A Variable Number of Members, Packing Version

An alternative approach is first to enumerate either all possible or all interesting feasible groups and then solve an optimization problem of the form:

Maximize value of the groups selected
subject to:
    Each object is in at most one of the selected groups.

The advantage of this formulation is, when it can be used, it typically can be solved more easily than the assignment formulation. The disadvantages are it may have a huge number of decision variables, especially if the typical group size is more than three. If there are n distinct objects, and all groups are of size k, then there are $n!/(k!\ (n-k)!)$ distinct groups. For example, if $n = 50$ and $k = 3$, then there are 19,600 candidate groups.

This formulation uses the idea of composite variables. This is frequently a useful approach for a problem for which the original or "natural" formulation is difficult to solve. Setting a particular composite variable to 1 represents setting a particular combination of the original variables to 1. We generate only those composite variables that correspond to feasible combinations of the original variables. This effectively eliminates many of the fractional solutions that would appear if one solved the LP relaxation of the original formulation. The composite variable idea is a form of what is sometimes called column generation. The path formulation in network models is also an example of the use of composite variables.

Example: Packing Financial Instruments, revisited.
The packing approach to formulating a model for this problem constructs all possible packages or groups that just satisfy the one million minimum. The general form of the LP/IP is:

Maximize value of packages selected
subject to:
Each object appears in at most one selected package.

In the formulation below, we will use sparse sets to represent our packages. We assume that we need not consider packages of more than four objects. An attractive feature of the packing/partitioning formulation is that we can easily attach an essentially arbitrary score to each possible group. In particular, the following formulation applies a squared penalty to the extent to which a package of loans exceeds the target of $1M.

```
MODEL:
! Object bundling model.  (OBJBUNDH);
! A broker has a number of loans of size from $55,000 to
$946,000.
 The broker would like to group the loans into packages
 so that each package has at least $1M in it, preferably
 not much more,
 and the number of packages is maximized;
! Keywords: bundling, financial, set packing;
 SETS:
 OBJECT: VALUE;
 ENDSETS
 DATA:
   OBJECT =  A   B   C   D   E   F   G   H   I   J   K  L   M   N   P   Q   R;
    VALUE = 910 870 810 640 550 250 120  95 55 200 321 492 567 837 193 364
946;
! The value in each bundle must be >= PKSIZE;
    PKSIZE = 1000;
 ENDDATA
 SETS:
 !Enumerate all 2,3, and 4 object unordered sets ≤ package
size;
  BNDL2( OBJECT, OBJECT) | &1 #LT# &2
      #AND# (VALUE(&1) + VALUE(&2)) #GE# PKSIZE:  X2, OVER2;
  BNDL3( OBJECT, OBJECT, OBJECT) | &1 #LT# &2 #AND#  &2 #LT#
&3
```

```
    #AND# ( VALUE(&1) + VALUE(&2) + VALUE(&3) #GE# PKSIZE):
      X3, OVER3;
  BNDL4( OBJECT, OBJECT, OBJECT, OBJECT) | &1 #LT# &2
    #AND#  &2 #LT# &3 #AND# &3 #LT# &4 #AND# (( VALUE(&1) +
    VALUE(&2) + VALUE(&3) + VALUE( &4)) #GE# PKSIZE):  X4,
OVER4;
 ENDSETS
!-----------------------------------------------;
!Compute the overage of each bundle;
  @FOR( BNDL2( I,J):
    OVER2(I,J) = VALUE(I) + VALUE(J) - PKSIZE;
      );
  @FOR( BNDL3( I,J,K):
    OVER3(I,J,K) = VALUE(I)+VALUE(J)+VALUE(K) - PKSIZE
      );
  @FOR( BNDL4( I,J,K,L):
    OVER4(I,J,K,L) = VALUE(I)+VALUE(J)+VALUE(K)+VALUE(L)-
PKSIZE;
      );

! Maximize score of packages assembled. Penalize a package
that
  is over the minimum package size;
  MAX= @SUM( BNDL2( I,J): X2(I,J) * (1-(OVER2(I,J)/PKSIZE)^2))
      +@SUM( BNDL3( I,J,K):
                    X3(I, J,K) * (1-(OVER3(I,J,K)/PKSIZE)^2))
     + @SUM( BNDL4( I,J,K,L):
                  X4(I,J,K,L) * (1-
(OVER4(I,J,K,L)/PKSIZE)^2));

  @FOR( OBJECT( M):
! Each object M can be in at most one of the selected bundles;
    @SUM( BNDL2( I, J)| I #EQ# M #OR# J #EQ# M: X2( I, J))
  + @SUM( BNDL3( I, J, K)| I #EQ# M #OR# J #EQ# M #OR# K #EQ#
M:
         X3( I, J, K))
  + @SUM( BNDL4( I, J, K, L)|
       I #EQ# M #OR# J #EQ# M #OR# K #EQ# M #OR# L #EQ# M:
         X4( I, J, K, L)) <= 1;
       );

! The X's must = 0 or 1;
  @FOR( BNDL2( I, J): @BIN( X2( I, J)););
  @FOR( BNDL3( I, J, K): @BIN( X3( I, J, K)););
  @FOR( BNDL4( I, J, K, L): @BIN( X4( I, J, K, L)););
END
```

```
Global optimal solution found at iteration:           19
Objective value:                              7.989192
                     Variable           Value
                    X2( A, H)        1.000000
                 OVER2( A, H)        5.000000
                    X2( B, P)        1.000000
                 OVER2( B, P)        63.00000
                    X2( C, F)        1.000000
                 OVER2( C, F)        60.00000
                    X2( D, Q)        1.000000
                 OVER2( D, Q)        4.000000
                    X2( E, L)        1.000000
                 OVER2( E, L)        42.00000
                    X2( I, R)        1.000000
                 OVER2( I, R)        1.000000
                    X2( J, N)        1.000000
                 OVER2( J, N)        37.00000
                   X3( G, K, M)      1.000000
               OVER3( G, K, M)       8.000000
```

Notice that this allocation is slightly more balanced than the previous solution based on the assignment formulation. The largest "overage" is 63,000 rather than 120,000. This is because the grouping formulation provided an easy way to penalize large packages.

11.7.6 Groups with A Variable Number of Members, Cutting Stock Problem

Another application in which the partitioning or packing approach has worked well is the cutting stock problem in the paper and steel industry. We revisit the example introduced in chapter 7. There we manually enumerated all possible patterns or packings drawn from 8 different finished good widths into each of three different raw material widths. The formulation below automatically enumerates all possible patterns. For each raw material width, the formulation automatically enumerates all possible groupings of 1, 2, …,7 finished good widths so that the sum of the finished good widths is less than or equal to the raw material width.

One notable feature of this formulation is that it introduces a shortcut that may be important in keeping computation time low when there are many, e.g., more than 20, objects. To illustrate the shortcut, consider the three declarations:

```
! Enumerate all possible cutting patterns with 1 fg;
  rxf(rm,fg)| lenf(&2) #le# lenr(&1): x1;
! Enumerate all possible patterns with 2 fg;
  rxf2( rxf, fg) |
     &2 #le# &3 #and# (lenf(&2) + lenf(&3) #le# lenr(&1)): x2;
! Enumerate all possible patterns with 3 fg;
  rxf3( rxf2, fg)| &3 #le# &4
      #and# (lenf(&2) + lenf(&3)+ lenf(&4) #le# lenr(&1)): x3;
```

The declaration `rxf(rm,fg),` by itself, tells LINGO to generate all combinations of one raw material and one finished good. The condition `| lenf(&2) #le# lenr(&1)` , however, tells

LINGO to not generate a combination of a raw material(the index &1) and finished good(index &2) for which the length(or width depending upon your orientation) of the finished good is greater than that of the raw material. So, for example, the combination (R36, F38) will not be a member of rxf. There will be four elements in rxf for which the first item of the pair is R36, namely (R36, F34), (R36, F24), (R36, F15), and (R36, F10).

Now consider how to generate all feasible combinations containing two finished good widths. The obvious declaration would be: rxf2(rm, fg, fg) | &2 #le# &3 #and# (lenf(&2) + lenf(&3) #le# lenr(&1))
The condition &2 #le# &3 says we do not care about the order of the finished goods in the pattern, so we might as well restrict ourselves to listing the finished goods in the pattern in sorted order. The condition lenf(&2) + lenf(&3) #le# lenr(&1) restricts the elements of set rxf2 to feasible ones. This declaration would be valid, but we did not do it. Why? Instead we used the declaration rxf2(rxf, fg). The latter was used mainly for computational reasons. With the latter, LINGO considers every combination of the elements of the set rxf and each finished good. Consider the case when the raw material is r36. If the declaration rxf2(rm, fg, fg) is used, then LINGO would look at 8 * 8 = 64 combinations of two finished goods and keep only the four combinations (r36, f24, f10), (r36, f15, f15), (r36, f15, f10), and (r36, f10, f10). If on the other hand, the declaration rxf2(rxf, fg) is used, then when the raw material is R36, LINGO will only consider 4*8 = 32 combinations. The 4 arises because set rxf contains only 4 elements for which the first member of the pair is R36. For sets rxf3, and higher, the computational savings can be even higher.

```
! Cutting stock solver(cutgent);
! Keywords: cutting stock;
SETS:
! Each raw material has a size(length) and quantity;
  rm: lenr, qr;
! Ditto for each finished good;
  fg: lenf, qf;
ENDSETS
DATA:
! Describe the raw materials available;
  rm, lenr, qr =
  R72  72   9999
  R45  48   9999
  R36  36   9999;
! Describe the finished goods needed;
  fg, lenf, qf =
  F60  60   500
  F56  56   400
  F42  42   300
  F38  38   450
  F34  34   350
  F24  24   100
  F15  15   800
  F10  10  1000;
ENDDATA
```

```
SETS:
! Enumerate all possible cutting patterns with 1 fg;
  rxf(rm,fg)| lenf(&2) #le# lenr(&1): x1;
! Enumerate all possible patterns with 2 fg;
  rxf2( rxf, fg) |
        &2 #le# &3 #and# (lenf(&2) + lenf(&3) #le# lenr(&1)):
x2;
! Enumerate all possible patterns with 3 fg;
  rxf3( rxf2, fg)| &3 #le# &4 #and#
               (lenf(&2) + lenf(&3)+ lenf(&4) #le# lenr(&1)):
x3;
! Enumerate all possible patterns with 4 fg;
  rxf4( rxf3, fg)| &4 #le# &5 #and#
     (lenf(&2) + lenf(&3) + lenf(&4)+lenf(&5) #le# lenr(&1)):
x4;
! Enumerate all possible patterns with 5 fg;
  rxf5( rxf4, fg)| &5 #le# &6 #and# (lenf(&2) + lenf(&3)+
lenf(&4)+lenf(&5)+lenf(&6)
          #le# lenr(&1)): x5;
! Enumerate all possible patterns with 6 fg;
  rxf6( rxf5, fg)| &6 #le# &7 #and# (lenf(&2) + lenf(&3)+
lenf(&4)+lenf(&5)
      +lenf(&6)+lenf(&7) #le# lenr(&1)): x6;

ENDSETS
! Minimize length of material used;

  MIN = @SUM( rxf(r,f1): lenr(r)*x1(r,f1))
      + @SUM( rxf2(r,f1,f2): lenr(r)*x2(r,f1,f2))
      + @SUM( rxf3(r,f1,f2,f3): lenr(r)*x3(r,f1,f2,f3))
      + @SUM( rxf4(r,f1,f2,f3,f4): lenr(r)*x4(r,f1,f2,f3,f4))
      + @SUM( rxf5(r,f1,f2,f3,f4,f5):
lenr(r)*x5(r,f1,f2,f3,f4,f5))
      + @SUM( rxf6(r,f1,f2,f3,f4,f5,f6):
lenr(r)*x6(r,f1,f2,f3,f4,f5,f6));

! We have to satisfy each finished good demand;
  @FOR( fg(f):
     @SUM(rxf(r,f): x1(r,f))
   + @SUM(rxf2(r,f1,f2)| f #eq# f1: x2(r,f1,f2))
   + @SUM(rxf2(r,f1,f2)| f #eq# f2: x2(r,f1,f2))
   + @SUM(rxf3(r,f1,f2,f3)| f #eq# f1: x3(r,f1,f2,f3))
   + @SUM(rxf3(r,f1,f2,f3)| f #eq# f2: x3(r,f1,f2,f3))
   + @SUM(rxf3(r,f1,f2,f3)| f #eq# f3: x3(r,f1,f2,f3))
   + @SUM(rxf4(r,f1,f2,f3,f4)| f #eq# f1: x4(r,f1,f2,f3,f4))
   + @SUM(rxf4(r,f1,f2,f3,f4)| f #eq# f2: x4(r,f1,f2,f3,f4))
   + @SUM(rxf4(r,f1,f2,f3,f4)| f #eq# f3: x4(r,f1,f2,f3,f4))
```

```
    + @SUM(rxf4(r,f1,f2,f3,f4)| f #eq# f4: x4(r,f1,f2,f3,f4))
    + @SUM(rxf5(r,f1,f2,f3,f4,f5)| f #eq# f1:
x5(r,f1,f2,f3,f4,f5))
    + @SUM(rxf5(r,f1,f2,f3,f4,f5)| f #eq# f2:
x5(r,f1,f2,f3,f4,f5))
    + @SUM(rxf5(r,f1,f2,f3,f4,f5)| f #eq# f3:
x5(r,f1,f2,f3,f4,f5))
    + @SUM(rxf5(r,f1,f2,f3,f4,f5)| f #eq# f4:
x5(r,f1,f2,f3,f4,f5))
    + @SUM(rxf5(r,f1,f2,f3,f4,f5)| f #eq# f5:
x5(r,f1,f2,f3,f4,f5))
    + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6)| f #eq# f1:
                                    x6(r,f1,f2,f3,f4,f5,f6))
    + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6)| f #eq# f2:
                                    x6(r,f1,f2,f3,f4,f5,f6))
    + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6)| f #eq# f3:
                                    x6(r,f1,f2,f3,f4,f5,f6))
    + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6)| f #eq# f4:
                                    x6(r,f1,f2,f3,f4,f5,f6))
    + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6)| f #eq# f5:
                                    x6(r,f1,f2,f3,f4,f5,f6))
    + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6)| f #eq# f6:
                                    x6(r,f1,f2,f3,f4,f5,f6))
     >= qf(f);
       );

! We cannot use more raw material than is available;
  @FOR( rm( r):
     @SUM(rxf(r,f): x1(r,f))
   + @SUM(rxf2(r,f1,f2): x2(r,f1,f2))
   + @SUM(rxf3(r,f1,f2,f3): x3(r,f1,f2,f3))
   + @SUM(rxf4(r,f1,f2,f3,f4): x4(r,f1,f2,f3,f4))
   + @SUM(rxf5(r,f1,f2,f3,f4,f5): x5(r,f1,f2,f3,f4,f5))
   + @SUM(rxf6(r,f1,f2,f3,f4,f5,f6): x6(r,f1,f2,f3,f4,f5,f6))
        <= qr(r);
     );

 ! Can only run integer quantities of each pattern;
     @FOR(rxf:  @GIN(x1));
     @FOR(rxf2: @GIN(x2));
     @FOR(rxf3: @GIN(x3));
     @FOR(rxf4: @GIN(x4));
     @FOR(rxf5: @GIN(x5));
     @FOR(rxf6: @GIN(x6));
```

If you click on LINGO | Generate menu item, to display the scaler version of the model, you can see that the constraint for the 56 inch width is(hopefully reassuringly):

```
   X2_R72_F56_F15 + X2_R72_F56_F10 + X1_R72_F56 >= 400 ;
```

When we click on the Solve icon we get the solution:

```
Global optimal solution found at iteration:       31
   Objective value:                            119832.0

                                Variable           Value
                      X2( R72, F60, F10)         500.0000
                      X2( R72, F56, F15)         400.0000
                      X2( R72, F38, F34)         350.0000
                 X3( R72, F42, F15, F15)         186.0000
                 X3( R72, F38, F24, F10)         100.0000
            X4( R72, F42, F10, F10, F10)         114.0000
            X4( R45, F15, F10, F10, F10)         2.000000
  X6( R72, F15, F15, F10, F10, F10, F10)         13.00000
```

11.7.7 Groups with A Variable Number of Members, Vehicle Routing

The following vehicle routing example demonstrates that you can in fact perform a little optimization computation as part of the column or group generation. The example we use is a variation of the vehicle routing problem considered in section 11.6.3. The first and major part of this model is devoted to enumerating all minimal feasible trips with at most seven stops. By feasible trip, we mean that the amount of material to be delivered to the stops in the trip does not exceed the vehicle capacity of 18 pallets. By minimal we mean that for a trip that visits a given set of stops, the trip visits the stops in a sequence that minimizes the distance traveled.

Given that all minimal feasible trips have been generated, the following simple integer program is solved:

Minimize Cost of trips selected;
Subject to:
For each stop:
Exactly one trip includes this stop.

This little example has 15 stops, so the integer program has 15 constraints, and a large number of 0/1 variables(about 7300 in fact) , equal in number to the number of minimal feasible trips.

The tricky part is how we generate the sets of minimal feasible trips, PSET2, PSET3, etc. and the assocated minimal distances, D2( ), D3( ), etc. To start understanding ideas, consider the variable `D4(I,J,K,L)`. It has the following definition.

`D4(I,J,K,L)` = minimum distance required to start at the depot, visit stops `I`, `J`, and `K` in any order and then visit stop `L`. If `DIST(I,J)` is the distance matrix, then Held and Karp(1962) observed that if `D3` is defined in similar fashion, then `D4` can be computed by the dynamic programming recursion:

```
D4(I,J,K,L) = min[D3(I,J,K)+DIST(K,L),
                  D3(I,K,J)+DIST(J,L),
                  D3(J,K,I)+DIST(I,L)]
```

The complete formulation follows.

```
MODEL:   ! (vrgenext);
```

```
! The Vehicle Routing Problem (VRP) occurs in many service
systems such as delivery, customer pick-up, repair and
maintenance. A fleet of vehicles, each with fixed
capacity, starts at a common depot and returns to the
depot after visiting locations where service is demanded.
This LINGO model generates all feasible one vehicle routes
and then chooses the least cost feasible multi-vehicle
combination;
SETS:
 CITY: Q;
 ! Q(I) = amount required at city I(given),
       must be delivered by just 1 vehicle;
 CXC( CITY, CITY): DIST;
 ! DIST(I,J) = distance from city I to city J;
ENDSETS
DATA:
CITY= Chi Den Frsn Hous  KC  LA Oakl Anah Peor Phnx Prtl Rvrs Sacr SLC Sntn SBrn;
! Amount to be delivered to each customer;
    Q= 0    6    3    7   7  18    4    5    2    6    7    2   4   3    3    2 ;
! city 1 represents the common depot, i.e. Q( 1) = 0;
! Distance from city I to city J is same(but need not be) from J to I;
DIST=  ! To City;
!Chi  Den Frsn Hous   KC   LA Oakl Anah Peor Phnx Prtl Rvrs Sacr  SLC Sntn SBrn From;
   0  996 2162 1067  499 2054 2134 2050  151 1713 2083 2005 2049 1390 1187 1996 ! Chcago;
 996    0 1167 1019  596 1059 1227 1055  904  792 1238 1010 1142  504  939 1001 ! Denver;
2162 1167    0 1747 1723  214  168  250 2070  598  745  268  162  814 1572  265 ! Fresno;
1067 1019 1747    0  710 1538 1904 1528  948 1149 2205 1484 1909 1438  197 1533 ! Huston;
 499  596 1723  710    0 1589 1827 1579  354 1214 1809 1535 1742 1086  759 1482 ! K-City;
2054 1059  214 1538 1589    0  371   36 1943  389  959   54  376  715 1363   59 ! L. A.;
2134 1227  168 1904 1827  371    0  407 2043  755  628  425   85  744 1729  422 ! Oaklnd;
2050 1055  250 1528 1579   36  407    0 1933  379  995   45  412  711 1353   55 ! Anahm;
 151  904 2070  948  354 1943 2043 1933    0 1568 2022 1889 1958 1299 1066 1887 ! Peoria;
1713  792  598 1149 1214  389  755  379 1568    0 1266  335  760  648  974  333 ! Phnix;
2083 1238  745 2205 1809  959  628  995 2022 1266    0 1001  583  767 2086  992 ! Prtlnd;
2005 1010  268 1484 1535   54  425   45 1889  335 1001    0  430  666 1309   10 ! Rvrsde;
2049 1142  162 1909 1742  376   85  412 1958  760  583  430    0  659 1734  427 ! Scrmto;
1390  504  814 1438 1086  715  744  711 1299  648  767  666  659    0 1319  657 ! SLC;
1187  939 1572  197  759 1363 1729 1353 1066  974 2086 1309 1734 1319    0 1307 ! SAnt;
1996 1001  265 1482 1533   59  422   55 1887  333  992   10  427  657 1307    0 ! SBrn;;
! VCAP is the capacity of a vehicle in 40"x48" pallets;
VCAP = 18;
ENDDATA
SETS:
! Enumerate all sets of various sizes of cities that are load
feasible;
 SET2(CITY,CITY)|&1 #GT# 1  #AND# &1 #LT# &2
              #AND# (Q(&1)+Q(&2)#LE# VCAP):;
 SET3(SET2,CITY)|&2 #LT# &3
              #AND# (Q(&1)+Q(&2)+Q(&3)#LE# VCAP):;
 SET4(SET3,CITY)|&3 #LT# &4
              #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)#LE# VCAP):;
 SET5(SET4,CITY)|&4 #LT# &5
              #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)+Q(&5)#LE# VCAP):;
 SET6(SET5,CITY)|&5 #LT# &6
              #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)+Q(&5)+Q(&6)#LE# VCAP):;
 SET7(SET6,CITY)|&6 #LT# &7
        #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)+Q(&5)+Q(&6)+Q(&7)#LE# VCAP):;
```

```
! Enumerate all partially ordered sets with a
       specific city as the last one;
 PSET2(CITY,CITY)| &1 #GT# 1 #AND# &1#NE#&2
                   #AND# (Q(&1)+Q(&2)#LE# VCAP): D2,X2;
 PSET3(SET2,CITY)| &1#NE#&3 #AND# &2#NE#&3
                   #AND# (Q(&1)+Q(&2)+Q(&3)#LE# VCAP): D3,X3;
 PSET4(SET3,CITY)| &1#NE#&4 #AND# &2#NE#&4 #AND# &3 #NE# &4
                  #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)#LE# VCAP): D4,X4;
 PSET5(SET4,CITY)| &1#NE#&5 #AND# &2#NE#&5 #AND# &3 #NE# &5
              #AND# &4 #NE# &5
             #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)+Q(&5)#LE# VCAP): D5,X5;
 PSET6(SET5,CITY)| &1#NE#&6 #AND#
        &2#NE#&6 #AND# &3 #NE# &6 #AND# &4 #NE# &6 #AND# &5 #NE# &6
        #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)+Q(&5)+Q(&6)#LE# VCAP): D6,X6;
 PSET7(SET6,CITY)| &1#NE#&7 #AND# &2#NE#&7 #AND# &3#NE#&7 #AND#
     &4#NE#&7 #AND# &5#NE#&7 #AND# &6#NE#&7
  #AND# (Q(&1)+Q(&2)+Q(&3)+Q(&4)+Q(&5)+Q(&6)+Q(&7)#LE# VCAP): D7,X7;
ENDSETS

! Compute shortest distance to visit all cities in PSET, and
   ending up at last city in each
   partially ordered set, using Held&Karp DP.
 Essential idea:
   DS( S,t) = minimum distance to visit all cities in
    S and then end up at t.  The recursion is:
   DS( S, t) = min{k in S: DS(S-k,k) + DIST(k,t)};
@FOR(PSET2(I,J):
  D2(I,J) = DIST(1,I) + DIST(I,J);
  @BIN(X2);
    );
@FOR(PSET3(I,J,K):
 ! @SMIN is the min of a list of scalers. D3(I,J,K) = min cost of
  starting at 1, visiting I and J in some order, and then K;
  D3(I,J,K) = @SMIN( D2(I,J) + DIST(J,K), D2(J,I) + DIST(I,K));
  @BIN(X3);
    );
@FOR(PSET4(I,J,K,L):
 !D4(I,J,K,L) = min cost of starting at 1, visiting I, J, & K
  in some order, and then L;
  D4(I,J,K,L) =
    @SMIN( D3(I,J,K)+DIST(K,L),
           D3(I,K,J)+DIST(J,L),
           D3(J,K,I)+DIST(I,L));
   @BIN( X4);
    );

@FOR(PSET5(I,J,K,L,M):
  D5(I,J,K,L,M) =
   @SMIN( D4(I,J,K,L)+DIST(L,M),
          D4(I,J,L,K)+DIST(K,M),
          D4(I,K,L,J)+DIST(J,M),
          D4(J,K,L,I)+DIST(I,M));
```

```
    @BIN(X5);
    );

@FOR(PSET6(I,J,K,L,M,N):
  D6(I,J,K,L,M,N) =
    @SMIN( D5(I,J,K,L,M)+DIST(M,N),
           D5(I,J,K,M,L)+DIST(L,N),
           D5(I,J,L,M,K)+DIST(K,N),
           D5(I,K,L,M,J)+DIST(J,N),
           D5(J,K,L,M,I)+DIST(I,N));
  @BIN(X6);
    );

@FOR(PSET7(I,J,K,L,M,N,P):
  D7(I,J,K,L,M,N,P) =
     @SMIN( D6(I,J,K,L,M,N)+DIST(N,P),
            D6(I,J,K,L,N,M)+DIST(M,P),
            D6(I,J,K,M,N,L)+DIST(L,P),
            D6(I,J,L,M,N,K)+DIST(K,P),
            D6(I,K,L,M,N,J)+DIST(J,P),
            D6(J,K,L,M,N,I)+DIST(I,P));
  @BIN(X7);
    );

! and finally, the optimization model...
 Min cost of routes chosen, over complete routes ending back at 1;
   Min =
    + @SUM( PSET2(I,J)| J #EQ# 1: D2(I,J)*X2(I,J))
    + @SUM( PSET3(I,J,K)| K #EQ# 1: D3(I,J,K)*X3(I,J,K))
    + @SUM( PSET4(I,J,K,L)| L #EQ# 1: D4(I,J,K,L)*X4(I,J,K,L))
    + @SUM( PSET5(I,J,K,L,M)| M #EQ# 1: D5(I,J,K,L,M)*X5(I,J,K,L,M))
    + @SUM( PSET6(I,J,K,L,M,N)| N #EQ# 1:
                                 D6(I,J,K,L,M,N)*X6(I,J,K,L,M,N))
    + @SUM( PSET7(I,J,K,L,M,N,P)| P #EQ# 1:
                                 D7(I,J,K,L,M,N,P)*X7(I,J,K,L,M,N,P));

! Each city must be on exactly one complete route;
 @FOR( CITY(I1)| I1 #GT# 1:
   + @SUM( PSET2(I,J) | J #EQ# 1 #AND# ( I #EQ# I1): X2(I,J))
   + @SUM( PSET3(I,J,K) |K #EQ# 1 #AND# (I#EQ#I1 #OR# J#EQ# I1):
             X3(I,J,K))
    + @SUM( PSET4(I,J,K,L) |L #EQ# 1 #AND#
         (I#EQ#I1 #OR# J#EQ# I1 #OR# K#EQ# I1): X4(I,J,K,L))
    + @SUM( PSET5(I,J,K,L,M) |M #EQ# 1 #AND#
         (I#EQ#I1 #OR# J#EQ# I1 #OR# K#EQ# I1 #OR# L#EQ#I1):
              X5(I,J,K,L,M))
    + @SUM( PSET6(I,J,K,L,M,N) |N #EQ# 1 #AND#
         (I#EQ#I1 #OR# J#EQ# I1 #OR# K#EQ# I1 #OR# L#EQ#I1
         #OR# M#EQ#I1): X6(I,J,K,L,M,N))
    + @SUM( PSET7(I,J,K,L,M,N,P) |P #EQ# 1 #AND#
         (I#EQ#I1 #OR# J#EQ#I1 #OR# K#EQ#I1 #OR# L#EQ#I1
         #OR# M#EQ#I1 #OR# N#EQ#I1)
       :  X7(I,J,K,L,M,N,P))  = 1;
```

```
        );
```

It takes about 4 seconds to get the following solution

```
Global optimal solution found at iteration:          134
   Objective value:                                17586.00

                                    Variable           Value
                                 X2( LA, CHI)        1.000000
                            X3( KC, PEOR, CHI)       1.000000
                     X4( DEN, HOUS, SNTN, CHI)       1.000000
              X5( FRSN, OAKL, PRTL, SACR, CHI)       1.000000
         X6( ANAH, PHNX, RVRS, SLC, SBRN, CHI)       1.000000
```

The obvious question is, how well does this formulation scale up? The final set partitioning integer program is not too challenging. The big challenge is generating and storing the possibly huge number of trips. A crucial consideration is the number of stops per trip. If this is small, e.g., three, then the number of trips will be manageable. A typical vehicle routing problem may have around 100 stops. The number of possible minimal distance trips, each of which visit three out of 100 stops, is 100!/[3!97!] = 161,700. This is a manageable number of variables for an efficient IP solver.

11.8 Linearizing Products of Variables

We have previously seen products of 0/1 variables, such as $y_1 \times y_2$ and y_1^2 can be represented by linear expressions by means of a simple transformation. This transformation generalizes to the case of the product of a 0/1 variable and a continuous variable.

To illustrate, suppose the product $x \times y$ appears in a model, where y is 0/1 while x is nonnegative continuous. We want to replace this nonlinear component by a (somewhat bigger) linear component. If we have an upper bound (M_x) on the value of x, an upper bound (M_y) on the product $x \times y$, and we define $P = x \times y$, then the following linear constraints will cause P to take on the correct value:

$$P \le x$$
$$P \le M_y \times y$$
$$P \ge x - M_x \times (1 - y)$$

Hanson and Martin (1990) show how this approach is useful in setting prices for products when we allow bundling of products. Bundle pricing is a form of quantity discounting. Examples of products that might be bundled are (a) airfare, hotel, rental car, tours, and meals or (b) computer, monitor, printer, and hard disk. Stigler (1963) showed how a movie distributor might improve profits by leasing bundles of movies rather than leasing individual movies. Bundling assumes it is easy for the seller to assemble the bundle and difficult for a buyer to unbundle. Otherwise, a reseller could buy the bundle at a discount and then sell the individual components at a markup.

11.8.1 Example: Bundling of Products

Microland Software has recently acquired an excellent word processing product to complement its own recently developed spreadsheet product. Microland is contemplating offering the combination of the two products for a discount. After demonstrating the products at a number of diverse professional meetings, Microland developed the following characterization of the market:

		Maximum Price Market Segment is Willing To Pay for Various Bundles		
Market Segment	**Size in 10,000**	**Spreadsheet Only**	**Wordprocessor Only**	**Both**
Business/ Scientific	7	450	110	530
Legal/ Administrative	5	75	430	480
Educational	6	290	250	410
Home	4.5	220	380	390

We will refer to each market segment as simply a "customer". Economic theory suggests a customer will buy the product that gives the greatest consumer surplus, where consumer surplus is defined as the price the customer is willing to pay for the product (the "reservation price") minus the market price for the product. For example, if the prices for the three bundles, spreadsheet only, word processor only, and both together, were set respectively at 400, 150, and 500, then the business/scientific market would buy the spreadsheet alone because the consumer surplus is 50 vs. –40 and 30 for the other two bundles.

To give a general model of this situation, define:

R_{ij} = reservation price of customer i for bundle j,
N_i = "size of customer" i (i.e., number of individual customers in segment i),
s_i = consumer surplus achieved by customer i,
y_{ij} = 1 if customer i buys bundle j, 0 otherwise,
x_j = price of bundle j set by the seller.

We will treat the empty bundle as just another bundle, so we can say every customer buys exactly one bundle.

The seller, Microland, would like to choose the x_j to:

$$\text{Maximize} \quad \sum_i \sum_j N_i y_{ij} x_j$$

The fact that each customer will buy exactly one bundle is enforced with:

For each customer i:

$$\sum_j y_{ij} = 1$$

For each customer i, its achieved consumer surplus is:

$$s_i = \sum_j (R_{ij} - x_j) y_{ij}$$

Customer i will buy only the bundle j for which its consumer surplus, s_i, is the maximum. This is enforced by the constraints:

For each customer i and bundle j:

$$s_i \geq R_{ij} - x_j$$

A difficulty with the objective function and the consumer surplus constraints is they involve the product $y_{ij}x_j$. Let us follow our previous little example and replace the product $y_{ij}x_j$ by P_{ij}. If M_j is an upper bound on x_j, then, proceeding as before, to enforce the definition $P_{ij} = y_{ij}x_j$, we need the constraints:

$$P_{ij} \leq x_j$$
$$P_{ij} \leq R_{ij}\, y_{ij}$$
$$P_{ij} \geq x_j - (1 - y_{ij})M_j.$$

Making these adjustments to the model, we get:

Maximize $\sum_i \sum_j N_i P_{ij}$

subject to:

For each customer i

$$\sum_j y_{ij} = 1;$$

For each customer i, bundle j:

$$s_i \geq R_{ij} - x_j;$$

For each customer i:

$$s_i = \sum_j (R_{ij} y_{ij} - P_{ij});$$

To enforce the nonlinear condition $P_{ij} = y_{ij}\, x_j$, we have for each i and j:

$$P_{ij} \leq x_j$$
$$P_{ij} \leq R_{ij}\, y_{ij}$$
$$P_{ij} \geq x_j - (1 - y_{ij})M_j.$$

For all i and j:

$$y_{ij} = 0 \text{ or } 1$$

In explicit form, the LINGO model is:

```
MODEL:
SETS:
 MARKET/B, L, E, H/:S, N;
 ITEM/NONE, SO, WO, BOTH/:X;
 MXI(MARKET, ITEM):R, Y, P;
ENDSETS
DATA:
 N = 7, 5, 6, 4.5;    ! Market size;
 R =   0 450 110 530 ! Reservation;
       0  75 430 480 !   prices;
       0 290 250 410
       0 220 380 390;
 M = 600; !Max price of any bundle;
ENDDATA
```

```
! Maximize our total revenue = price * market size.
  P(i,j) = price customer i pays for product or item j,
           if i buys j, else = 0;
 MAX = @SUM(MXI(I, J): P(I, J) * N(I));
!Make the pick variables 0/1;
 @FOR(MXI:@BIN(Y));
!Each customer or market i picks or buys exactly one bundle;
 @FOR(MARKET(I): @SUM(ITEM(J): Y(I, J)) = 1);
! Each customer i's  achieved surplus, S(i), must be at
 least as good as from every possible bundle;
 @FOR(ITEM(I): @FOR(MARKET(J):
      S(I) >= R(I,J) - X(J)));
!Customer i's achieved surplus = reservations price
  of item purchased - its price;
 @FOR(MARKET(I):
   S(I) =  @SUM(ITEM(J): R(I, J) * Y(I, J)- P(I, J))
    ) ;
! Each price variable Pij must be.. ;
!   <= Xj , (the published price);
!   <= Rij * Yij (less than reservation price if bought);
!   >= Xj - M + M * Yij ;
 @FOR(MXI(I, J): P(I, J) <= X(J);
            P(I, J) <= Y(I, J) * R(I, J);
            R(I, J) >= X(J) - M + M * Y(I, J););
! Price of bundle should be <= sum of component prices;
 X( @INDEX(BOTH)) <= X(@INDEX(SO)) + X(@INDEX(WO));
! Price of bundle should be >= any one component;
 X(@INDEX(BOTH)) >= X(@INDEX(SO)); X(@INDEX(BOTH)) >= X(@INDEX(WO));
END
```

For the Microland problem, the solution is to set the following prices:

	Spreadsheet Only	Word Processing Only	Both
Bundle Price:	410	380	410

Thus, the business, legal and educational markets will buy the bundle of both products. The home market will buy only the word processor. Total revenues obtained by Microland are 90,900,000. The interested reader may show that, if bundling is not possible, then the highest revenue that Microland can achieve is only 67,150,000.

11.9 Representing Logical Conditions

For some applications, it may be convenient, perhaps even logical, to state requirements using logical expressions. A logical variable can take on only the values TRUE or FALSE. Likewise, a logical expression involving logical variables can take on only the values TRUE or FALSE. There are two major logical operators, #AND# and #OR#, that are useful in logical expressions.

The logical expression:

A #AND# B

is TRUE if and only if both A and B are true.

The logical expression:

A #OR# B

is TRUE if and only if at least one of A and B is true.

It is sometimes useful also to have the logical operator implication ($\Rightarrow$) written as follows:

$A \Rightarrow B$

with the meaning that if A is true, then B must be true.

Logical variables are trivially representable by binary variables with:

TRUE being represented by 1, and
FALSE being represented by 0.

If A, B, and C are 0/1 variables, then the following constraint combinations can be used to represent the various fundamental logical expressions:

Logical Expression	Mathematical Constraints
$C = A$ #AND# B	$C \le A$ $C \le B$ $C \ge A + B - 1$
$C = A$ #OR# B	$C \ge A$ $C \ge B$ $C \le A + B$
$A \Rightarrow C$	$A \le C$

11.10 Problems

1. The following problem is known as a segregated storage problem. A feed processor has various amounts of four different commodities, which must be stored in seven different silos. Each silo can contain at most one commodity. Associated with each commodity and silo combination is a loading cost. Each silo has a finite capacity, so some commodities may have to be split over several silos. For a similar problem arising in the loading of fuel tank trucks at Mobil Oil Company, see Brown, Ellis, Graves, and Ronen (1987). The following table contains the data for this problem.

	Loading Cost per Ton							
	Silo							Amount of Commodity To Be Stored
Commodity	1	2	3	4	5	6	7	
A	\$1	\$2	\$2	\$3	\$4	\$5	\$5	75 tons
B	2	3	3	3	1	5	5	50 tons
C	4	4	3	2	1	5	5	25 tons
D	1	1	2	2	3	5	5	80 tons
Silo Capacity in Tons	25	25	40	60	80	100	100	

a) Present a formulation for solving this class of problems.

b) Find the minimum cost solution for this particular example.

c) How would your formulation change if additionally there was a fixed cost associated with each silo that is incurred if anything is stored in the silo?

2. You are the scheduling coordinator for a small, growing airline. You must schedule exactly one flight out of Chicago to each of the following cities: Atlanta, Los Angeles, New York, and Peoria. The available departure slots are 8 A.M., 10 A.M., and 12 noon. Your airline has only two departure lounges, so at most two flights can be scheduled per slot. Demand data suggest the following expected profit contribution per flight as a function of departure time:

Expected Profit Contribution in $1000's

	Time		
Destination	**8**	**10**	**12**
Atlanta	10	9	8.5
Los Angeles	11	10.5	9.5
New York	17	16	15
Peoria	6.4	2.5	–1

Formulate a model for solving this problem.

3. A problem faced by an electrical utility each day is that of deciding which generators to start up at which hour based on the forecast demand for electricity each hour. This problem is also known as the unit commitment problem. The utility in question has three generators with the following characteristics:

Generator	Fixed Startup Cost	Fixed Cost per Period of Operation	Cost per Period per Megawatt Used	Maximum Capacity in Megawatts Each Period
A	3000	700	5	2100
B	2000	800	4	1800
C	1000	900	7	3000

There are two periods in a day and the number of megawatts needed in the first period is 2900. The second period requires 3900 megawatts. A generator started in the first period may be used in the second period without incurring an additional startup cost. All major generators (e.g., *A*, *B*, and *C* above) are turned off at the end of each day.

a) First, assume fixed costs are zero and thus can be disregarded. What are the decision variables?

b) Give the LP formulation for the case where fixed costs are zero.

c) Now, take into account the fixed costs. What are the additional (zero/one) variables to define?

d) What additional terms should be added to the objective function? What additional constraints should be added?

4. *Crude Integer Programming*. Recently, the U.S. Government began to sell crude oil from its Naval Petroleum Reserve in sealed bid auctions. There are typically six commodities or products to be sold in the auction, corresponding to the crude oil at the six major production and shipping points. A "bid package" from a potential buyer consists of (a) a number indicating an upper limit on how many barrels (bbl.) the buyer is willing to buy overall in this auction and (b) any number of "product bids". Each product bid consists of a product name and three numbers representing, respectively, the bid price per barrel of this product, the minimum acceptable quantity of this product at this price, and the maximum acceptable quantity of this product at this price. Not all product bids of a buyer need be successful. The government usually places an arbitrary upper limit (e.g., 20%) on the percentage of the total number of barrels over all six products one firm is allowed to purchase.

   To illustrate the principal ideas, let us simplify slightly and suppose there are only two supply sources/products, which are denoted by *A* and *B*. There are 17,000 bbls. available at *A* while *B* has 13,000. Also, there are only two bidders, the Mobon and the Exxil companies. The government arbitrarily decides either one can purchase at most 65% of the total available crude. The two bid packages are as follows:

Mobon:

Maximum desired = 16,000 bbls. total.

Product	Bid per Barrel	Minimum Barrels Accepted	Maximum Barrels Wanted
A	43	9000	16,000
B	51	6000	12,000

Exxil:

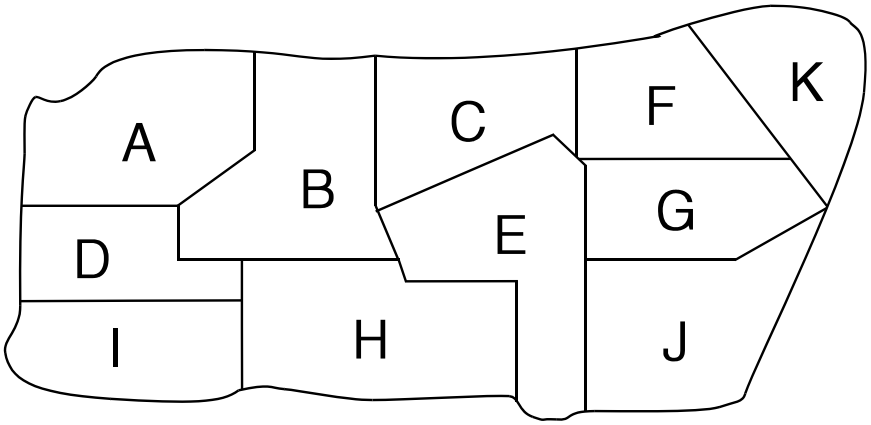
Maximum desired = No limit.

Product	Bid per Barrel	Minimum Barrels Accepted	Maximum Barrels Wanted
A	47	5000	10,000
B	50	5000	10,000

Formulate and solve an appropriate IP for the seller.

5. A certain state allows a restricted form of branch banking. Specifically, a bank can do business in county *i* if the bank has a "principal place of business" in county *i* or in a county sharing a nonzero-length border with county *i*. Figure 11.10 is a map of the state in question:

Figure 11.10 Districts in a State



Formulate the problem of locating a minimum number of principal places of business in the state, so a bank can do business in every county in the state. If the problem is formulated as a covering problem, how many rows and columns will it have? What is an optimal solution? Which formulation is tighter: set covering or simple plant location?

6. *Data Set Allocation Problem*. There are 10 datasets or files, each of which is to be allocated to 1 of 3 identical disk storage devices. A disk storage device has 885 cylinders of capacity. Within a storage device, a dataset will be assigned to a contiguous set of cylinders. Dataset sizes and interactions between datasets are shown in the table below. Two datasets with high interaction rates should not be assigned to the same device. For example, if datasets *C* and *E* are assigned to the same disk, then an interaction cost of 46 is incurred. If they are assigned to different disks, there is no interaction cost between *C* and *E*.

Dataset for Interaction (Seek Transition) Rates

	A	B	C	D	E	F	G	H	I	J	Dataset Size in Cylinders
A											110
B	43										238
C	120	10									425
D	57	111	188								338
E	96	78	46	88							55
F	83	58	421	60	63						391
G	77	198	207	109	73	74					267
H	31	50	43	47	51	21	88				105
I	38	69	55	21	36	391	47	96			256
J	212	91	84	53	71	40	37	35	221		64
											2249

Find an assignment of datasets to disks, so total interaction cost is minimized and no disk capacity is exceeded.

7. The game or puzzle of mastermind pits two players, a "coder" and a "decoder", against each other. The game is played with a pegboard and a large number of colored pegs. The pegboard has an array of 4 × 12 holes. For our purposes, we assume there are only six colors: red, blue, clear, purple, gold, and green. Each peg has only one color. The coder starts the game by selecting four pegs and arranging them in a fixed order, all out of sight of the decoder. This ordering remains fixed throughout the game and is appropriately called the code. At each play of the game, the decoder tries to match the coder's ordering by placing four pegs in a row on the board. The coder then provides two pieces of information about how close the decoder's latest guess is to the coder's order:

    1) The number of pegs in the correct position (i.e., color matching the coder's peg in that position), and
    2) The maximum number of pegs that would be in correct position if the decoder were allowed to permute the ordering of the decoder's latest guess.

    Call these two numbers *m* and *n*. The object of the decoder is to discover the code in a minimum number of plays.

The decoder may find the following IP of interest.

```
MAX  = XRED1;
  XRED1 + XBLUE1 + XCLEAR1 + XPURP1 + XGOLD1
      + XGREEN1 = 1;
  XRED2 + XBLUE2 + XCLEAR2 + XPURP2 + XGOLD2
      + XGREEN2 = 1;
  XRED3 + XBLUE3 + XCLEAR3 + XPURP3 + XGOLD3
      + XGREEN3 = 1;
  XRED4 + XBLUE4 + XCLEAR4 + XPURP4 + XGOLD4
      + XGREEN4 = 1;
 XRED1 + XRED2  + XRED3 + XRED4 - RED = 0;
 XBLUE1 + XBLUE2  + XBLUE3 + XBLUE4 - BLUE = 0;
 XCLEAR1 + XCLEAR2 + XCLEAR3 + XCLEAR4 - CLEAR = 0;
 XPURP1 + XPURP2 + XPURP3 + XPURP4 - PURP = 0;
 XGOLD1 + XGOLD2 + XGOLD3 + XGOLD4 - GOLD = 0;
 XGREEN1 + XGREEN2 + XGREEN3 + XGREEN4 - GREEN = 0;
END
```

All variables are required to be integer. The interpretation of the variables is as follows. *XRED1* = 1 if a red peg is in position 1, otherwise 0, etc.; *XGREEN4* = 1 if a green peg is in position 4, otherwise 0. Rows 2 through 5 enforce the requirement that exactly one peg be placed in each position. Rows 6 through 11 are simply accounting constraints, which count the number of pegs of each color. For example, *RED* = the number of red pegs in any position 1 through 4. The objective is unimportant. All variables are (implicitly) required to be nonnegative.

At each play of the game, the decoder can add new constraints to this IP to record the information gained. Any feasible solution to the current formulation is a reasonable guess for the next play. An interesting question is what constraints can be added at each play.

To illustrate, suppose the decoder guesses the solution *XBLUE1* = *XBLUE2* = *XBLUE3* = *XRED4* = 1, and the coder responds with the information that $m = 1$ and $m - n = 1$. That is, one peg is in the correct position and, if permutations were allowed, at most two pegs would be in the correct position. What constraints can be added to the IP to incorporate the new information?

8. The Mathematical Football League (MFL) is composed of M teams (M is even). In a season of $2(M - 1)$ consecutive Sundays, each team will play $(2M - 1)$ games. Each team must play each other team twice, once at home and once at the other team's home stadium. Each Sunday, k games from the MFL are televised. We are given a matrix $\{v_{ij}\}$ where v_{ij} is the viewing audience on a given Sunday if a game between teams i and j playing at team j's stadium is televised.

   a) Formulate a model for generating a schedule for the MFL that maximizes the viewing audience over the entire season. Assume viewing audiences are additive.

   b) Are some values of k easier to accommodate than others? How?

9. The typical automobile has close to two dozen electric motors. However, if you examine these motors, you will see that only about a half dozen distinct motor types are used. For inventory and maintenance reasons, the automobile manufacturer would like to use as few distinct types as possible. For cost, quality, and weight reasons, one would like to use as many distinct motor types as possible, so the most appropriate motor can be applied to each application. The table below describes the design possibilities for a certain automobile:

24-Month Failure Probability

Application	Number Required	Motor type A	B	C	D	E
Head lamps	2	0.002	0.01		0.01	0.007
Radiator fan	2		0.01	0.002		0.004
Wipers	2				0.007	
Seat	4	0.003			0.006	0.008
Mirrors	2			0.004	0.001	
Heater fan	1		0.006	0.001		
Sun roof	1	0.002			0.003	0.009
Windows	4	0.004	0.008	0.005		
Antenna	1	0.003		0.003	0.002	
	Weight	2	3	1.5	1	4
	Cost per Motor	24	20	36	28	39

For example, two motors are required to operate the headlamps. If type *D* motors are used for headlamps, then the estimated probability of a headlamp motor failure in two years is about 0.01. If no entry appears for a particular combination of motor type and application, it means the motor type is inappropriate for that application (e.g., because of size).

Formulate a solvable linear integer program for deciding which motor type to use for each application, so at most 3 motor types are used, the total weight of the motors used is at most 36, total cost of motors used is at most 585, and probability of any failure in two years is approximately minimized.

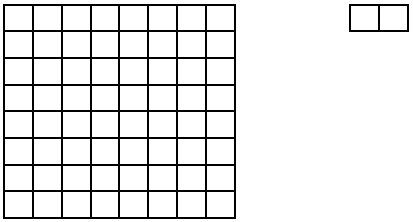
10. We have a rectangular three-dimensional container that is $30 \times 50 \times 50$. We want to pack in it rectangular three-dimensional boxes of the three different sizes: (a) $5 \times 5 \times 10$, (b) $5 \times 10 \times 10$, and (c) $5 \times 15 \times 25$.

A particular packing of boxes into the container is undominated if there is no other packing that contains at least as many of each of the three box types and strictly more of one of the box types.

Show there are no more than 3101 undominated packings.

11. Given the following:

Checkerboard and *domino*



If two opposite corners of the checkerboard are made unavailable, prove there is no way of exactly covering the remaining grid with 31 dominoes.

12. Which of the following requirements could be represented exactly with linear constraints? (You are allowed to use transformations if you wish.)

    (a) $(3 \times x + 4 \times y)/(2 \times x + 3 \times y) \leq 12$;
    (b) MAX $(x, y) < 8$;
    (c) $3 \times x + 4 \times y \times y \geq 11$; where y is 0 or 1;
    (d) ABS $(10 - x) \leq 7$ (Note ABS means absolute value);
    (e) MIN $(x, y) < 12$.

13. A common way of controlling access in many systems, such as information systems or the military, is with priority levels. Each user i is assigned a clearance level U_i. Each object j is assigned a security level L_j. A user i does not have access to object j if the security level of j is higher than the clearance level of i. Given a set of users; and, for each user, a list of objects to which that user does not to have access; and a list of objects to which the user should have access, can we assign U_i's and L_j's, so these access rights and denials are satisfied? Formulate as an integer program.

14. One of the big consumption items in the U.S. is automotive fuel. Any petroleum distributor who can deliver this fuel reliably and efficiently to the hundreds of filling stations in a typical distribution region has a competitive advantage. This distribution problem is complicated by the fact that a typical customer (i.e., filling station) requires three major products: premium gasoline, an intermediate octane grade (e.g., "Silver"), and regular gasoline. A typical situation is described below. A delivery tank truck has four compartments with capacities in liters of 13,600, 11,200, 10,800, and 4400. We would like to load the truck according to the following limits:

	Liters of		
	Premium	**Intermediate**	**Regular**
At least:	8,800	12,000	12,800
At most:	13,200	17,200	16,400

Only one gasoline type can be stored per compartment in the delivery vehicle. Subject to the previous considerations, we would like to maximize the amount of fuel loaded on the truck.

    (a) Define the decision variables you would use in formulating this problem as an IP.
    (b) Give a formulation of this problem.
    (c) What allocation do you recommend?

15. Most lotteries are of the form:

    Choose n numbers (e.g., $n = 6$) from the set of numbers $\{1, 2, ..., m\}$ (e.g., $m = 54$).

    You win the grand prize if you buy a ticket and choose a set of n numbers identical to the n numbers eventually chosen by lottery management. Smaller prizes are awarded to people who match k of the n numbers. For $n = 6$, typical values for k are 4 and 5. Consider a modest little lottery with $m = 7$, $n = 3$, and $k = 2$. How many tickets would you have to buy to guarantee winning a prize? Can you set this up as a grouping/covering problem?

16. A recent marketing phenomenon is the apparent oxymoron, "mass customization". The basic idea is to allow each customer to design his/her own product, and yet do it on an efficient, high-volume scale. A crucial component of the process is to automate the final design step involving the customer. As an example, IBM and Blockbuster recently announced a plan to offer "on-demand" production of customized music products at retail stores. Each store would carry an electronic "master" for every music piece a customer might want. The physical copy for the customer would then be produced for the customer while they wait. This opens up all manners of opportunities for highly customized musical products. Each customer might provide a list of songs to be placed on an audiocassette. A design issue when placing songs on a two-sided medium such as a cassette is how to allocate songs to sides. A reasonable rule is to distribute the songs, so the playing times on the two sides are as close to equal as possible. For an automatic tape player, this will minimize the "dead time" when switching from one side to another. As an example, we mention that Willie Nelson has recorded the following ten songs in duets with other performers:

Song	Time (min:secs)	Other Performer
1) Pancho and Lefty	4:45	Merle Haggard
2) Slow Movin Outlaw	3:35	Lacy J. Dalton
3) Are There any More Real Cowboys	3:03	Neil Young
4) I Told a Lie to My Heart	2:52	Hank Williams
5) Texas on a Saturday Night	2:42	Mel Tillis
6) Seven Spanish Angels	3:50	Ray Charles
7) To All the Girls I've Loved Before	3:30	Julio Iglesias
8) They All Went to Mexico	4:45	Carlos Santana
9) Honky Tonk Women	3:30	Leon Russell
10) Half a Man	3:02	George Jones

You want to collect these songs on a two-sided tape cassette album to be called "Half Nelson."

(a) Formulate and solve an integer program for minimizing the dead time on the shorter side.

(b) What are some of the marketing issues of allowing the customer to decide which song goes on which side?

17. Bill Bolt is hosting a party for his daughter Lydia on the occasion of her becoming of college age. He has reserved a banquet room with 18 tables at the Racquet Club on Saturday night. Each table can accommodate at most 8 people. A total of 140 young people are coming, 76 young men and 64 young ladies. Lydia and her mother, Jane, would like to have the sexes as evenly distributed as possible at the tables. They want to have at least 4 men and at least 3 women at each table.

    (a) Is it possible to have an allocation satisfying the above as well as the restriction there be at most 4 men at each table?

    (b) Provide a good allocation of the sexes to the tables.

18. The game or puzzle of Clue is played with a deck of 21 cards. At the beginning of a game, three of the cards are randomly selected and placed face down in the center of the table. The remaining cards are distributed face down as evenly as possible to the players. Each player may look at his or her own cards. The object of the game is to correctly guess the three cards in the center. At each player's turn, the player is allowed to either guess the identity of the three cards in the center or ask any other player a question of the form: "Do you have any of the following three cards?" (The asking player then publicly lists the three cards.) If the asked player has one of the three identified cards, then the asked player must show one of the cards to the asking player (and only to the asking player). Otherwise, the asked player simply responds "No". If a player correctly guesses the three cards in the center, then that player wins. If a player incorrectly guesses the three cards in the center, the player is out of the game.

    Deductions about the identity of various cards can be made if we define:

    $X(i, j) = 1$ if player i has card j, else 0.

    Arbitrarily define the three cards in the center as player 1. Thus, we can initially start with the constraints:

$$\sum_{j=1}^{21} X(1, j) = 3.$$

    For each card, $j = 1, 2, \ldots, 21$:

$$\sum_{i} X(i, j) = 1.$$

    (a) Suppose player 3 is asked: "Do you have either card 4, 8, or 17?" and player 3 responds "No." What constraint can be added?

    (b) Suppose in response to your question in (a), player 3 shows you card 17. What constraint can be added?

    (c) What LP would you solve in order to determine whether card 4 must be one of the cards in the center?

    Note, in the "implementation" of the game marketed in North America, the 21 cards are actually divided into three types: (i) six suspect cards with names like "Miss Scarlet," (ii) six weapons cards with names like "Revolver," and (c) nine room cards with names like "Kitchen." This has essentially no effect on our analysis above.